Production in Advance in Monetary Economies:
Random Matching and Bargaining*

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Abstract

Search-theoretic models of money assume that sellers can produce any amount of goods on demand. However, a majority of goods are produced in advance by sellers. In this paper we propose a new model of monetary search in which sellers produce partially durable goods in advance and then strive to find customers. This has two implications. First, sellers have the option to produce for themselves (home production) or for customers in exchange for money (market production). Second, both sides of the market take a risk: buyers invest in money which depreciates because of inflation, sellers produce perishable output in advance and none is sure to trade. We obtain three types of equilibria. Type I Equilibrium is a non-monetary pure-strategy equilibrium in which the seller produces only for himself and nothing for the market. Type II Equilibrium is a pure-strategy monetary equilibrium in which the seller produces, sells a fraction and keeps the leftover for himself. Type III Equilibrium has two parts, both of which are mixed-strategy monetary equilibria. In the first one buyers bring a unique amount of money and sellers randomize between two levels of output. In the second, the buyer brings multiple amounts of money with positive probabilities and likewise the seller brings multiple amounts of output with positive probabilities. Thus, with some probability this equilibrium gives rise to the buyer bringing more money than he spends.

Keywords: Money, Production in Advance, Perishability, Bargaining, Price Posting, Inflation

JEL Classification:
1 Introduction

Recent developments in monetary theory have provided explicit microfoundations to the circulation of money and to alternative payment arrangements such as credit. Applications range from the study of credit markets, optimal monetary policy, the relationship between inflation and unemployment or business cycles analysis.\(^1\) A typical assumption in this literature is that the amount of goods traded is produced on demand by sellers contingent on meeting an appropriate buyer. If no such meeting happens, sellers simply carry on to the next period with no production cost and no output. This assumption of production on demand, interpreted as the good being a service, was made initially in the second generation of monetary search models (Shi 1995, Trejos Wright, 1995) to suit the pairwise random matching-Nash bargaining framework in which those models were developed (see Kiyotaki and Wright 1989,1991,1993).

While this assumption proved very helpful, casual observation suggests that a vast majority of goods are produced in advance by sellers; that is, sellers produce goods prior to meeting a buyer. In this paper we propose a new model of monetary search in which sellers produce goods in advance and then strive to find customers. Our goal is to study the equilibrium features of such economy and associated economic policy recommendations.

Besides being more realistic, such a model brings to the forefront two key features of exchange. First, in models with production on demand, buyers bear all the risks from trade. Not only do they invest in monetary units that depreciate over time, but they have no guarantee that they will be able to trade them for goods. Consequently such models put excessive weight on buyers’ money holding decision: by deciding ex ante on their real balances and therefore on ex post output, buyers’ real balances decisions end up driving the entire economy. By having sellers producing before they meet with buyers, our model restores balance between the two sides of the market. Buyers take risk investing in money, and sellers take risk by producing output in advance.

A second feature that we think is central to exchange economies is the degree to which

\(^1\)See Williamson and Wright (2010a,2010b) for a presentation of this literature.
goods are perishable, or can be turned into alternative output or use if not sold. This is clearly a non-issue in production on demand models. But once sellers produce prior to meeting a buyer, one must address the issue of unsold output. In our model we will assume that sellers have access to a technology that enable them to turn any output they have not been able to sell on the frictional market into a different kind of good that they can either sell on a Walrasian market at the equilibrium market price, or consume. As a result sellers always have the option to produce and consume their own output (whether intend it to or not). We call this feature of the model home production by contrast to market production which corresponds to output produced in order to be sold. As for the assumption that sellers produce in advance, we think that this partial transformability assumption (or partial perishability depending on how we interpret it) restores balance between buyers and seller: buyers invest in depreciating money, sellers produce perishable goods. We emphasize that both production on demand and perishability are complementary assumptions.

In this paper we build a monetary economy with production in advance and assume that agents meet randomly and bargain.\(^2\) In this environment sellers will form expectations on buyers’ money holdings, and sellers will form expectations on sellers’ output. We show that there are three types of equilibria: type I equilibrium is purely home production as buyers bring no money to the market and sellers produce for their own consumption next period. This non-monetary autarkic equilibrium is characterized by high nominal interest rate. Type II equilibrium is similar to Lagos and Wright (2005) in the sense that all sellers produce a unique quantity and all buyers bring the same amount of real balances to the market. Buyers and sellers trade when they meet, and sellers take some leftover home. In Type III, sellers randomize over the output level and buyers may also randomize over their money holding. Hence it is possible for buyers to spend less money than what they brought to the market. Type III equilibrium appears when the nominal interest rate is low. Given a low nominal interest rate, a buyer strategically brings an amount of money to the market that gives him a chance to get more

\(^2\)In a companion paper Anbarci, Dutu and Sun (2010) we study a similar economy in which sellers produce and post terms of trade in advance.
output from a seller when output turns out to be high. In parallel, given the high amount of real balances brought by the buyer, a seller is indifferent between producing less and selling all output to a buyer in exchange for his money and producing more and taking some leftover home. That is, sellers randomize over the output level at low nominal interest rates. This kind of strategic interactions between buyers and sellers can only emerge under our production in advance assumption.

The article is organized as follows. Section 2 presents the model. In section 3 we characterize the equilibrium and comparative statics of a benchmark model in between Lagos and Wright (2005) and our model: while goods are partly durable sellers still produce on demand in the frictional market rather than in advance in the Walrasian market. Section 4 presents the model with production in advance and section 5 concludes.

2 The basic model

Time is discrete and goes on forever. Each period is divided into two trading subperiods. In the first subperiod agents participate in a centralized Walrasian market where they can produce and consume any quantity of a first good, called good 1. Then they enter a second frictional market where they trade a second type of good, good 2. While good 1 is produced and consumed during the Walrasian market, good 2 is produced once the Walrasian market is closed but before the frictional market opens. That is, good 2 is produced in advance. Both goods are fully divisible at the production and the consumption stage. We use $\beta$ to denote the discount factor between the Walrasian market and the frictional market. There is no discounting between the frictional market and the Walrasian market.

There is a continuum of anonymous, infinitely lived agents who, following Rocheteau and Wright (2005), differ in terms of when they produce and consume the goods. In the first subperiod, i.e. in the centralized market, all agents can produce and consume good 1. In the second subperiod, i.e. during the frictional market, agents are divided into buyers who want to consume the good but could not produce it, and sellers who can produce it but cannot
consume it. This assumption generates a temporal double coincidence problem. Combined with the assumption that good 1 is at least partially perishable (no commodity money) and that agents are anonymous (no credit), this makes money essential for trade. The measure of buyers is set to $1$ and the measure of sellers is set to $n$. In the frictional market buyers get to trade with probability $\alpha(n)$ and sellers get to trade with probability $\frac{\alpha(n)}{n}$. We will assume that $\alpha'(n) > 0$, $\alpha''(n) < 0$, $\alpha(0) = 0$ and $\alpha(\infty) = 1$. We will start with $n$ exogenous and then endogenous by adding an entry cost of $k$ for sellers.

In the first submarket, consuming $\bar{x}$ units of good 1 yields utility $v(\bar{x})$ with $v'(x) > 0$ and $v''(x) < 0$. Producing $\bar{x}$ units of good 1 costs $\bar{x}$. Once the Walrasian market has closed, sellers can produce any quantity $\bar{q}$ of good 2 for the frictional market to come. Total costs of producing $\bar{q}$ are given by $c(\bar{q})$ with $c(0) = c'(0) = 0$ and $c''(\bar{q}) > 0$. We note that $\bar{q} \leq \hat{q}$ the quantity purchased by a buyer, which is then consumed immediately and yields the buyer utility $u(\bar{q})$ with $u(0) = 0$, $u'(0) > 1 - \delta$, $u'(q) > 0$, and $u''(q) < 0$. We assume that the goods produced for the frictional market are perishable at rate $\delta \in (0, 1)$.

If there is any unsold output on this market, $\bar{q} - \hat{q}$, then a fraction $1 - \delta$ of it is carried over to the next period Walrasian market where it is transformed into good 1 according to

$$y = (1 - \delta) (\bar{q} - \hat{q})$$

where $y$ is the production of good 1 coming from the transformation of good 2. If $\delta = 1$ then good 2 is perfectly durable and any unsold good 2 is turned one-to-one into good 1. If $\delta = 0$ then good 2 is 100% perishable and any unsold output is just lost. Note that while good 2 is partially perishable, good 1 is fully perishable. One implication of this is that no good whether good 1 or good 2 survives beyond the Walrasian market, ruling out the possibility of commodity money. The instantaneous utility function of a buyer is

$$U^b = v(\bar{x}) - \bar{x} + \beta u(\bar{q})$$

and that of a seller is

$$U^s = v(\bar{x}) - \bar{x} - \beta c(\bar{q}) .$$
Money in this economy is a perfectly divisible and storable object whose value relies on its use as a medium of exchange. Each period new money is injected or withdrawn via lump-sum transfers by the central bank at rate $\tau$ such that $M_{t+1} = (1 + \tau) M_t$ in which $M_t$ is the quantity of money available at time $t$. Only buyers receive this transfer. Inflation is forecasted perfectly and both the quantity theory and the Fisher effect apply: if the money supply increases at rate $\tau$, so do prices and the nominal interest rate. Denoting $r$ the real interest rate, since $\beta = 1/(1 + r)$ the Fisher equation $(1 + i_t) = (1 + r)(1 + \pi_t)$ enables to write the nominal interest rate as $i_t = (1 - \beta + \tau_t)/\beta$ where $\pi_t$ denotes the inflation level at $t$.

The nominal price of good 1 on the centralized market is normalized to 1 and the price of money in terms of good 1, noted $\phi_t$, adjusts to match supply and demand. That is, 1 unit of money buys $\phi_t$ units of the good on the centralized market, or 1 unit of the good costs $1/\phi_t$ units of money. Note that we will focus on steady state equilibria where the aggregate real money supply is constant. Thus, $\phi = \phi_{t+1} (1 + \tau)$ where the subscript $+1$ denotes the value of a variable (or value function) in the next period. By analogy to good 1 and good 2, we denote $\hat{m}$ the quantity of money held by a buyer and $\hat{m}$ the quantity spend. The sequence of events is represented below.

### 2.1 Sellers

The state variable for a seller include the amount of good 2 and the amount of money he is holding, noted $q$ and $m$ respectively. Let $W^s(q, m)$ be the value function for a seller in the
centralized market. Let \( V^s(q) \) be the value function of a seller holding \( q \) units of good 2 in the frictional market where the absence of any monetary state variable in \( V^s \) reflects the fact that sellers have no incentive bring money to the frictional market. In the Walrasian market a seller’s problem is then

\[
W^s(q, m) = \max_{\hat{x}, \bar{x}, \hat{q}} \left\{ v(\hat{x}) - \bar{x} - c(\hat{q}) + \beta V^s(\hat{q}) \right\},
\]

s.t. \( \hat{x} = \phi m + (1 - \delta) q + \bar{x} \).  

A seller chooses the optimal amount of good 2 to produce and bring to the frictional market, \( \hat{q} \), for which he suffers disutility \( c(\hat{q}) \). He also chooses the optimal amount of good 1 to produce and consume, \( \bar{x} \) and \( \hat{x} \) respectively. His budget constraint ensures that the amount of money received from buyers, \( m \), plus the leftover from the previous market \((1 - \delta) q\) and the production of good 1, \( \bar{x} \), enables him to buy \( \hat{x} \). Substituting out for \( \bar{x} \) yields

\[
W^s(q, m) = \max_{\hat{x}, \bar{x}, \hat{q}} \left\{ v(\hat{x}) - \hat{x} + \phi m + (1 - \delta) q - c(\hat{q}) + \beta V^s(\hat{q}) \right\}.
\]

In the frictional market a seller trades with a buyer with probability \( \frac{\alpha(n)}{n} \) in which case he receives \( \hat{m} \) units of money in exchange for \( \hat{q} \) units of good 2 and proceeds with \( \hat{q} - \hat{q} \) units of goods 2. With probability \( \left(1 - \frac{\alpha(n)}{n}\right) \) she does not trade and proceeds with no money and her entire stock of good 2. Her value function is then given by

\[
V^s(q) = \frac{\alpha(n)}{n} W^s_{+1}[\hat{q} - \hat{q}, \hat{m}(\hat{q})] + \left(1 - \frac{\alpha(n)}{n}\right) W^s_{+1}(\hat{q}, 0).
\]

### 2.2 Buyers

Since a buyer consumes immediately any quantity of good 1 in the centralized market or good 2 he purchases in the frictional market, the only state variable for a buyer is his money holding, noted \( m \). Let \( W^b(m) \) be the value function for a buyer in the centralized market. Let \( V^b(\hat{m}) \) be the value function of a buyer holding \( \hat{m} \) units of money in the frictional market. We have

\[
W^b(m) = \max_{\hat{x}, \bar{x}, \hat{m}} \left\{ v(\hat{x}) - \bar{x} - c(\hat{q}) + \beta V^b(\hat{m}) \right\},
\]

s.t. \( \phi \hat{m} + \hat{x} = \phi (m + T) + \bar{x} \).
where \( \tilde{m} \) is the money taken out of this market to the frictional market. Other than that the interpretation is similar to (2). Substituting out for \( \tilde{x} \) yields

\[
W^b(m) = \max_{\tilde{x}, \tilde{m}} \left\{ v(\tilde{x}) - \tilde{x} + \phi(m + T) - \phi\tilde{m} + \beta V^b(\tilde{m}) \right\}
\]  

(6)

In the frictional market a buyer trades with probability \( \alpha(n) \) in which case he pays \( \tilde{m} \), consumes \( \hat{q} \) and proceeds with \( \tilde{m} - \hat{m} \) units of money. With probability \( 1 - \alpha(n) \) he does not trade and moves on to the centralized market with an unchanged stock of money. Therefore

\[
V^b(\tilde{m}) = \alpha(n) \left\{ u[\hat{q}(\tilde{m})] + W^b_{+1}[\tilde{m} - \hat{m}(\tilde{q})] \right\} + (1 - \alpha(n)) W^b_{+1}(\tilde{m}) .
\]  

(7)

In (7) the notations \( \tilde{m} = \hat{m}(\tilde{q}) \) and \( \hat{q} = \hat{q}(\tilde{m}) \) emphasize that in general the price paid depends on the quantity purchased, and vice versa, via the pricing mechanism.

2.3 Production in advance by sellers

Sellers must decide before the opening of the frictional market how much of their production good to bring along. By contrast to traditional production-on-demand models, sellers are taking a risk similar to that taken by buyers: they are not sure they will be able to trade this output, and if they don’t, it is going to be of lesser value next period because of goods’ perishability.

To derive the seller’s choice of output, first note that next period’s value function for a seller who trades this period is given by

\[
W^s_{+1}[\tilde{q} - \hat{q}, \hat{m}(\tilde{q})] = v(\hat{x}^*) - \hat{x}^* + \phi_{+1} \hat{m}(\hat{q}) + (1 - \delta)(\hat{q} - \tilde{q}) + \max_{\tilde{q}} \left\{ -c(\tilde{q}) + \beta V^s(\tilde{q}) \right\}
\]  

(8)

where \( \tilde{q} \) represents the choice of output for the next period given that \( \hat{q} \) was chosen for this one. Similarly, the next period’s value function for a seller who does not trade this period is given by

\[
W^s_{+1}(\tilde{q}, 0) = v(\hat{x}^*) - \hat{x}^* + (1 - \delta) \hat{q} + \max_{\tilde{q}} \left\{ -c(\tilde{q}) + \beta V^s(\tilde{q}) \right\} .
\]  

(9)

Once (8) and (9) are inserted into (4) one obtains

\[
V^s(\tilde{q}) = v(\hat{x}^*) - \hat{x}^* + \frac{\alpha(n)}{n} \left\{ \phi_{+1} \hat{m}(\hat{q}) + (1 - \delta)(\hat{q} - \tilde{q}) \right\} + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \hat{q} + \max_{\tilde{q}} \left\{ -c(\tilde{q}) + \beta V^s(\tilde{q}) \right\} .
\]  

(10)
Finally, inserting (10) into (3) and getting rid of constant terms, the seller’s program is

$$\max_{\hat{q} \geq 0} \Phi(\hat{q}) = -c(\hat{q}) + \beta \left( \frac{\alpha(n)}{n} [\phi_{+1} \hat{m}(\hat{q}, \tilde{m}) + (1 - \delta) (\hat{q} - q(\hat{q}, \tilde{m}))] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \hat{q} \right).$$

(11)

When deciding on his production of good 2, the seller maximizes the difference between his production costs, which are sunk, and the discounted expected return from selling part of it with probability $\frac{\alpha(n)}{n}$, or selling none of it with probability $1 - \frac{\alpha(n)}{n}$. In both cases only a fraction $1 - \delta$ of the leftover is carried forward to the next centralized market.

### 2.4 Cash in advance by buyers

Buyers must decide during the Walrasian market how much money to bring to the frictional market. As sellers do, buyers too take a risk as they are not certain to trade. And if they do not trade their money holdings will be less valuable due to the central bank lump sum transfers. To derive the buyer’s choice of money, first note that the next period’s value function for a buyer who trades this period is given by

$$W^b_{+1} [\tilde{m} - \hat{m}(\hat{q})] = v(\hat{x}^*) - \hat{x}^* + \phi_{+1} (\tilde{m} - \hat{m}(\hat{q})) + \max_{\hat{m}} \left\{ -\phi \hat{m} + \beta V^b(\hat{m}) \right\}$$

(12)

where $\hat{m}$ represents the choice of money for the next period given that $\tilde{m}$ was chosen for this one. Similarly, next period’s value function for a buyer who does not trade this period is given by

$$W^b_{+1} (\hat{m}) = v(\hat{x}^*) - \hat{x}^* + \phi_{+1} (\tilde{m} + T) + \max_{\hat{m}} \left\{ -\phi \hat{m} + \beta V^b(\hat{m}) \right\}.$$  

(13)

If we insert those value functions into (7) one obtains

$$V^b(\hat{m}) = v(\hat{x}^*) - \hat{x}^* + \phi_{+1} T + \alpha(n) \left\{ u[\hat{q}(\hat{m})] + \phi_{+1} [\tilde{m} - \hat{m}(\hat{q})] \right\} + \left[ 1 - \alpha(n) \right] \phi_{+1} \hat{m} + \max_{\hat{m}} \left\{ -\phi \hat{m} + \beta V^b(\hat{m}) \right\}.$$  

(14)

By inserting (14) into (6) and getting rid of constant terms, the buyer’s program simplifies into

$$\max_{\tilde{m} \geq 0} \Psi(\tilde{m}) = -\phi \tilde{m} + \beta \left\{ \alpha(n) \left\{ u[\hat{q}(\hat{q}, \tilde{m})] + \phi_{+1} [\tilde{m} - \hat{m}(\hat{q}, \tilde{m})] \right\} + \left[ 1 - \alpha(n) \right] \phi_{+1} \tilde{m} \right\}.$$  

(15)

When choosing their money holding for the frictional market to come, buyers maximize the difference between the opportunity cost of money and the discounted expected return from
spending part of it with probability $\alpha(n)$ and getting some utility in return or proceeding to the next centralized market with the same amount of money in nominal value—but less real balances since $\phi_{t+1} = \frac{\phi}{1+\tau}$, which happens with probability $1 - \alpha(n)$.

Note that while buyers have no incentive to bring more money than what they intend to spend, sellers may have an incentive to bring more goods than what they intend to sell. To see that note than while good 1 can be produced directly in the centralized market, good 1 can also be produced indirectly from good 2 leftovers according to the technology given by $y = (1 - \delta) (\hat{q} - \check{q})$, with associated production cost of $c(y)$. Given that the two technologies bear different costs, sellers will find it profitable to produce more good 2, denoted $\check{q}$, than what they intend to sell, denoted $\hat{q}$ with $\check{q} < \hat{q}$, if the production cost of $\check{q} - \hat{q}$ is less via the indirect technology than via the direct technology.

3 A benchmark model: production on the spot with perishable goods

Before we proceed any further, it will be important to note that, in this section and the remainder of this paper, we assume that upon entering the frictional market, buyers and sellers are matched randomly in which case terms of trade are determined by bilateral bargaining. Negotiation in a two-party relationship creates a surplus that would otherwise be unavailable to any of the parties. Any negotiation involves a bargaining process in which players believe that there is the possibility of reaching a unanimous agreement, although there is a conflict of interest. In addition, the adoption of any agreement requires the approval of both parties, whether it is a binding agreement – à la the Nash (1950) solution – or an equilibrium outcome – à la Rubinstein (1982) alternating-offers bargaining scheme; by definition, in any equilibrium, no party would unilaterally deviate from it. It has been shown by Binmore, Rubinstein and Wolinsky (1986) that the Rubinstein scheme converges to a possibly asymmetric (generalized) Nash bargaining solution (Kalai, 1977). Thus, we presume here that buyers and sellers adhere to the generalized Nash solution outcome as soon as they meet in a match with their relative bargaining power. Bargaining weights are assumed to be constant and equal to $\theta$ for a buyer
and $1 - \theta$ for a seller.

In this section, before we proceed to our main model which is in the next section, we present a benchmark model in which goods are perishable but sellers produce on demand (i.e., on the spot) in the frictional market rather than in advance in the Walrasian market. The sequence of events is then the following: first buyers receive a lump-sum payment from the central bank and then decide on their money holdings. Then buyers and sellers meet randomly in the frictional market. When a buyer and seller meet they first bargain over terms of trade (the quantity traded $\hat{q}$ and the price paid by the buyer $\hat{m}$). Then the seller produces. Note that a seller might want to produce more that what he is going to sell $\hat{q}$ if he intends to carry over some of it to the coming centralized market. As a matter of fact the marginal return on home production is given by how much of good 2 will be available in the next centralized market, that is $1 - \delta$ and the marginal cost is given by $c'(q)$. As a result a seller will always produce at least $\hat{q}$ given by

$$c'(\hat{q}) = 1 - \delta.$$  

If bargaining results in the buyer purchasing $\hat{q} < \hat{q}$ then the seller is going to keep $\hat{q} - \hat{q}$ for himself. If bargaining results in the buyer purchasing $\hat{q} > \hat{q}$ then the seller simply produces $\hat{q}$ and does not keep anything for himself.

A seller’s value function in the centralized market is given by

$$W_s^s(q, m) = \max_{\hat{x}, \hat{\hat{q}}} \left\{ v(\hat{x}) - \hat{x} + \beta V_s \right\},$$

s.t. $\hat{x} = \phi m + (1 - \delta) q + \hat{x}$

so that

$$W_s(q, m) = \max_{\hat{x}, \hat{\hat{q}}} \left\{ v(\hat{x}) - \hat{x} + \phi m + (1 - \delta) q + \beta V_s \right\}. \quad (17)$$

If $\hat{q} < \hat{q}$ a seller’s value function in the frictional market is given by

$$V_s^s(\hat{q}) = \max_{\hat{q}} -c(\hat{q}) + \frac{\alpha(n)}{n} W_s^{s+1} [\hat{q} - \hat{q}, \hat{m}(\hat{q})] + \left(1 - \frac{\alpha(n)}{n}\right) W_s^{s+1} (\hat{q}, 0) \quad (18)$$

and the Nash product is

$$\max_{\hat{q}, \hat{\hat{q}} \leq \hat{m}} \left[ u(\hat{q}) + W_s^{b+1} [\hat{m} - \hat{m}] - W_s^{b+1} (\hat{m}) \right]^{\theta} \left[ W_s^{b+1} [\hat{q}_1 - \hat{q}, \hat{m}] - W_s^{b+1} (\hat{q}_2, 0) \right]^{1-\theta}$$

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which simplifies into
\[
\max_{q, \hat{m} \leq \hat{m}} \left[ u(\hat{q}) - \phi_{+1} \hat{m} \right]^\theta \left[ \phi_{+1} \hat{m} - (1 - \delta) \hat{q} \right]^{1-\theta}.
\]
If \( \hat{q} > \bar{q} \) a seller’s value function in the frictional market is given by
\[
V^s(q) = \frac{\alpha(n)}{n} \left\{ -c(\hat{q}) + W^s_{+1} [0, \hat{m}(\hat{q})] \right\} + \left( 1 - \frac{\alpha(n)}{n} \right) \max_{\hat{q}} \left\{ -c(\hat{q}) + W^s_{+1} (\bar{q}, 0) \right\}
\]
and the Nash product is
\[
\max_{q, \hat{m} \leq \hat{m}} \left[ u(\hat{q}) + W^b_{+1}[\hat{m} - \hat{m}] - W^b_{+1}(\hat{m}) \right]^\theta \left\{ \{-c(\hat{q}) + W^s_{+1}[0, \hat{m}]\} - \{-c(\hat{q}) + W^s_{+1}(\bar{q}, 0)\} \right\}^{1-\theta}
\]
which simplifies into
\[
\max_{q, \hat{m} \leq \hat{m}} \left[ u(\hat{q}) - \phi_{+1} \hat{m} \right]^\theta \left[ \phi_{+1} \hat{m} - (1 - \delta) \hat{q} - \{c(\hat{q}) - c(\bar{q})\} \right]^{1-\theta}.
\]
Equations are unchanged for buyers.

**Proposition 1** The possible equilibria, represented in Fig. 2, are as follows:

**Type I [market and home production]:** Sellers produce \( \bar{q} \) and sell \( \bar{q} < \hat{q} \) to buyers with
\[
\frac{c'(\bar{q})}{c'(\bar{q})} = \beta (1 - \delta) \quad \frac{u'(\bar{q})}{g_1'(\bar{q})} = 1 + \frac{i}{\alpha(n)}
\]

**Type II [market production only]:** Sellers produce \( \bar{q} > \bar{q} \) given by
\[
\frac{u'(\bar{q})}{g_2'(\bar{q})} = 1 + \frac{i}{\alpha(n)}
\]
and sell it altogether to buyers.

When goods durability is high sellers will always produce a certain amount of good 2 for themselves in order to turn it into good 1. Although the seller looses a fraction \( \delta \) of that good, he is more than compensated by the low production costs he enjoys in the frictional market. For intermediate durability the nominal interest rate starts playing a critical role. When holding money does not cost much sellers prefer to sell their entire output to the market reallocating resources from home production to money. As inflation increases, however, sellers will shift back to home production as it becomes more profitable than getting money through market exchange.
4 The main model: production in advance with perishable goods

Given production in advance with perishable goods, the generalized Nash solution to the bargaining between a buyer and a seller is given by

\[
\operatorname{arg\ max}_{\tilde{q} \leq q, \tilde{m} \leq m} \mathcal{B}(\tilde{q}, \tilde{m}) = \left[ u(\tilde{q}) + W_{+1}^b [\tilde{m} - \tilde{m}] - W_{+1}^b (\tilde{m}) \right]^\theta \left[ W_{+1}^s [\tilde{q} - \tilde{q}, \tilde{m}] - W_{+1}^s (\tilde{q}, 0) \right]^{1-\theta}
\]

in which \( W_{+1}^b (\tilde{m}) \) and \( W_{+1}^s (\tilde{q}, 0) \) are the buyer’s and seller’s threat payoffs, respectively. Using the same simplification techniques as in Section 2.3 and 2.4, this simplifies into

\[
\operatorname{max}_{\tilde{q} \leq q, \tilde{m} \leq m} \mathcal{B}(\tilde{q}, \tilde{m}) = \left[ u(\tilde{q}) - \phi_{+1} \tilde{m} \right]^\theta \left[ \phi_{+1} \tilde{m} - (1 - \delta) \tilde{q} \right]^{1-\theta}.
\]

Let us define the following functions:

\[
\begin{align*}
g(x) &\equiv \frac{(1 - \delta)(1 - \theta)}{\theta u'(x) + (1 - \theta)(1 - \delta)} u(x) + \frac{\theta u'(x)}{\theta u'(x) + (1 - \theta)(1 - \delta)} (1 - \delta) x \\
h(x) &\equiv (1 - \theta) u(x) + \theta (1 - \delta) x \\
q_N &\equiv u^{-1}(1 - \delta) \\
m_N &\equiv h(q_N)/\phi_{+1} = g(q_N)/\phi_{+1}
\end{align*}
\]

The function \( g \) and \( h \) correspond to the first-order conditions of the bargaining problem with respect to \( \tilde{q} \) and \( \tilde{m} \) respectively. In the Nash bargaining process, given that the buyer
brings \( \hat{m} \) to the frictional market, the seller needs to produce \( g^{-1}(\phi_{+1}\hat{m}) = \hat{q} \). Likewise, in the Nash bargaining process, given that the seller brings \( \hat{q} \) to the frictional market, the buyer needs to bring \( h(\hat{q})/\phi_{+1} = \hat{m} \). The intersection of \( g(q) \) and \( h(q) \) yields \( q_N \) and \( m_N \). That is, \((q_N, m_N)\) represents the unconstrained solution. Denote by \( (\hat{q}(\hat{q}, \hat{m}), \hat{m}(\hat{q}, \hat{m})) = \arg \max_{\hat{q}\leq \hat{q}, \hat{m}\leq \hat{m}} B(\hat{q}, \hat{m}) \).

**Lemma 1** \((\hat{q}(\hat{q}, \hat{m}), \hat{m}(\hat{q}, \hat{m})) = (\min\{g^{-1}(\phi_{+1}\hat{m}), q_N, \hat{q}\}, \min\{h(\hat{q})/\phi_{+1}, m_N, \hat{m}\})\).

In Lagos and Wright, the buyer and seller who meet decide on the spot on output and gains from trade are shared according to bargaining weights. Given the inefficiencies, \( \alpha (n) < 1 \) [frictional matching], \( \beta < 1 \) [discounting] and \( \tau > 0 \) [inflation], buyers do not bring real balances that can buy the efficient quantity \( q^* \) that sellers could produce, unless the central bank neutralizes those inefficiencies by restoring a zero rate of return on money. All of this is perfectly forecasted by buyers in forming their pre-market best-responses. Thus, buyers actually decide on the size of production and sellers are the passive party in that respect in Lagos and Wright. That is, sellers simply react once they meet a buyer, without any making any decision before going to the market for a match.

In our setting, things work out very differently. Here, a seller does not decide on the spot how much to produce; he produces in advance instead. Therefore, both parties are involved in decision making before going to the frictional market for a potential match. To the above inefficiencies one must now add production in advance and goods’ perishability.

Let \( \zeta \) solves the following equation:

\[
\frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha (n)}.
\]

In the following, we assume \( u'(x)/g'(x) \) is strictly decreasing on \([0, q_N]\); hence \( \zeta \) is unique whenever it exists. The strict monotonicity of \( u'(x)/g'(x) \) guarantees the uniqueness of the Nash equilibrium. If \( u'(x)/g'(x) \) is not strictly decreasing, then multiple equilibria may exist, but the essential features of the equilibria remain the same. A sufficient condition for \( u'(x)/g'(x) \) to be
strictly decreasing is provided in the Appendix. Denote by

\[ q_H = c^{-1}(\beta (1 - \delta)), \]
\[ q_L = c^{-1}(\beta \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta)). \]

Note that \( q_L \) is what the seller would produce if constrained to exchange the entire quantity when he meets the buyer at the frictional market. On the other hand, \( q_H \) is what the seller would produce if he does not face any such constraint.

**Proposition 2 (Type I Equilibrium)** If \( u(0) \leq 1 + \frac{i}{\alpha(n)}, \) or equivalently, \( u(0) \leq (1 + \frac{i}{\alpha(n)}) (1 - \delta) \), then \((\bar{q}, \bar{m}) = (q_H, 0)\) is the unique Nash equilibrium.

If \( \bar{q} \leq q_N \), then it can be readily seen that \( q_H \) maximizes the seller’s payoff for any \( \bar{m} \geq 0 \). Suppose now \( \bar{q} < q_N \). If \( \bar{m} = g(\bar{q})/\phi + 1 \), then \( q_H \) again maximizes the seller’s payoff. Let \( \bar{m}_C = (g(\bar{q})/\phi + 1, g(q_H)/\phi + 1) \) be such that \( \Phi(\max\{\bar{q}, h^{-1}(\phi + 1 \bar{m}_C)\}|\bar{m} = \bar{m}_C) = \Phi(q_H|\bar{m} = \bar{m}_C) \) whenever it exists, otherwise let \( \bar{m}_C = m_N \).

Type I Equilibrium is a pure-strategy equilibrium in which the marginal utility of \( q \) is low for the buyer. This leads to a non-market outcome where the seller produces only for himself and nothing for the market.

**Proposition 3 (Type II Equilibrium)** If \( u(0)/g(0) > 1 + \frac{i}{\alpha(n)} \) and \( \zeta \leq g^{-1}(\phi + 1 \bar{m}_C) \), then \((\bar{q}, \bar{m})\) specified below is the unique Nash equilibrium

\[
\begin{align*}
\bar{q} &= q_H, \\
\frac{u'(\bar{q}|\bar{m})}{g'(\bar{q}|\bar{m})} &= \frac{u'(g^{-1}(\phi + 1 \bar{m}_C))}{g'(g^{-1}(\phi + 1 \bar{m}_C))} = \frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha(n)}
\end{align*}
\]

Type II Equilibrium is a pure-strategy equilibrium in which the marginal utility of \( q \) is sufficiently high for the buyer. Out of three types of equilibria that arise in our model with Production in Advance, this equilibrium is the most comparable to that in Lagos and Wright (2005). In this equilibrium, the seller produces both for the market and for himself; i.e., even after he meets a buyer at the frictional market and exchanges the amount the buyer would like
to buy, the seller still is able to take some $q$ to the Walrasian market. The latter feature is an artifact of our model with Production in Advance. It does not account in Lagos and Wright (2005) since the producer cannot produce for himself (this is because the goods are not assumed to be durable in their model). Figure 2 partition the set of parameters in terms of Type II and Type III equilibria, that is assuming $u'(0) > 1 + \frac{i}{\alpha(n)}$.

Next, we consider mixed strategy equilibria. Let $\mu_b : B_{\mathbb{R}_+} \rightarrow [0, 1]$ (Borel probability measure on $\mathbb{R}_+$) denote the buyer’s strategy, and $\mu_s : B_{\mathbb{R}_+} \rightarrow [0, 1]$ the seller’s strategy, where $B_{\mathbb{R}_+}$ stands for the Borel $\sigma$–algebra in $\mathbb{R}_+$. Denote by $\mathcal{F}_b : \mathbb{R}_+ \rightarrow [0, 1]$ the distribution function induced by the buyer’s mixed strategy $\mu_b$, and $\mathcal{F}_s : \mathbb{R}_+ \rightarrow [0, 1]$ the distribution function induced by the seller’s mixed strategy $\mu_s$. Define $l(\mathcal{F}_i) = \sup\{x \in \mathbb{R}_+ : \mathcal{F}_i(x) = 0\}$ and $u(\mathcal{F}_i) = \inf\{x \in \mathbb{R}_+ : \mathcal{F}_i(x) = 1\}$, where $i \in \{b, s\}$. Define

$$\hat{m} = \frac{1}{\phi_{+1}} g(q_H) - \int_{q_L}^{q_H} \frac{c'(x) - \beta(1 - \frac{\alpha(n)}{n}) (1 - \delta)}{\beta \frac{\alpha(n)}{n} (1 - \delta)} \, dx.$$
Proposition 4 (Type III Equilibrium) Suppose \( \frac{u'(0)}{g'(0)} > 1 + \frac{i}{\alpha(n)} \) and \( \zeta > g^{-1}(\phi_{+1}\tilde{m}C) \).

(i) If \( \tilde{q}_L \geq h^{-1}(g(q_H)) \), then the pair \((\mu_s, \mu_b)\) constructed below constitutes a unique Nash equilibrium:

\[
\begin{align*}
\mu_b(\tilde{m}) &= \begin{cases} 
1 & \tilde{m} = \tilde{\tilde{m}} \\
0 & \tilde{m} \neq \tilde{\tilde{m}}
\end{cases} \\
\mu_s(\tilde{q}) &= \begin{cases} 
(1 + \frac{i}{\alpha(n)}) \frac{g'(g^{-1}(\phi_{+1}\tilde{m}))}{u'(g^{-1}(\phi_{+1}\tilde{m}))} & \tilde{q} = \tilde{q}_H \\
1 - (1 + \frac{i}{\alpha(n)}) \frac{g'(g^{-1}(\phi_{+1}\tilde{m}))}{u'(g^{-1}(\phi_{+1}\tilde{m}))} & \tilde{q} = \tilde{q}_L
\end{cases}
\end{align*}
\]

(ii) If \( \tilde{q}_L < h^{-1}(g(q_H)) \), then there could be multiple equilibria. All of them satisfy the following properties:

(a) \( \tilde{q}_L \leq l(F_s) < u(F_s) \leq \tilde{q}_H \) and \( g(\tilde{q}_L)/\phi_{+1} \leq l(F_b) \leq u(F_b) \leq g(\tilde{q}_H)/\phi_{+1} \).

(b) \( \mu_b \) is at most trinary.

(c) \( \mu_s \) is at least binary, and at most quaternary.

Type III Equilibrium has two parts, both of which are mixed-strategy equilibria, in which the marginal utility of \( q \) is very high to the buyer. In the first equilibrium, when the buyer...
brings a lot of money to the frictional market, the seller produces very little and when the buyer brings very little money to the frictional market, the seller produces much more for himself. Neither is optimal for the buyer. In the first case, the buyer would not want to bring that much money, and in the latter case, for $\bar{q}$ produced, the buyer would like bring more money to the frictional market. In this equilibrium, the buyer brings a moderate amount of money to the frictional market and the seller randomizes between the two output levels.

In the second mixed-strategy equilibrium, the buyer brings multiple amounts of money with positive probabilities and likewise the seller brings multiple amounts of output with positive probabilities. Thus, with some probability this equilibrium gives rise to the case where the buyer ends up bringing more money to the frictional market than he gives to the seller in exchange for $\bar{q}$ the seller brings to market. In both if these mixed-strategy equilibria, the seller produces some amount for the buyer and some amount for himself (i.e., as in Types I and II equilibria, here too, even after he meets a buyer at the frictional market and exchanges the amount the buyer would like to buy, the seller still is able to take some $q$ to the Walrasian market).

Appendix

A.0. Proof of Proposition 1

Type I [market and home production]: $\bar{q} < \bar{q}$.

We set up the Lagrangian

$$L = [u(\hat{q}) - \phi_{+1}\hat{m}]^\theta [\phi_{+1}\hat{m} - (1 - \delta)\hat{q}]^{1-\theta} + \lambda_1 (\bar{m} - \hat{m}).$$

The first-order condition with respect to $\hat{q}$ yields

$$\theta u'(\hat{q}) [\phi_{+1}\hat{m} - (1 - \delta)\hat{q}] = (1 - \theta) (1 - \delta) [u(\hat{q}) - \phi_{+1}\hat{m}]^\theta$$

and the first-order condition on $\hat{m}$ yields

$$(1 - \theta) \phi_{+1} [u(\hat{q}) - \phi_{+1}\hat{m}] = \theta \phi_{+1} [\phi_{+1}\hat{m} - (1 - \delta)\hat{q}] + \lambda_1 [u(\hat{q}) - \phi_{+1}\hat{m}]^{1-\theta} [\phi_{+1}\hat{m} - (1 - \delta)\hat{q}]^\theta.$$
$m = h(q) / \phi + 1$

$\Phi(q \mid m_0)$

$\Phi(q \mid m_1)$

$\Phi(q \mid m_2)$

$\Phi$ [Diagram]

$\Phi$ [Diagram]

$q_L$ $q_H$

$q_L$ $q_H$

$\Phi(q \mid m)$

FIGURE 3: The Shifts of $\Phi$ w.r.t. $m$
As in Lagos and Wright (2005) it is easy to show that buyers do not bring more than what they intend to spend so that \( \hat{m} = \bar{m}. \) Setting \( \hat{m} = \bar{m} \) into (20) one extracts

\[
\phi_{+1} \hat{m} = g_I [\hat{q} (\bar{m})] = \frac{(1 - \theta)(1 - \delta) u(\hat{q}) + \theta u'(\hat{q}) (1 - \delta) \hat{q}}{\theta u'(\hat{q}) + (1 - \theta)(1 - \delta)}.
\] (22)

The buyer’s problem is given by

\[
\max_{\hat{m}} -\phi \hat{m} + \beta \left\{ \alpha (n) \{u(\hat{q}(\bar{m}))\} + [1 - \alpha (n)] \phi_{+1} \bar{m} \right\}.
\]

Substituting out \( \phi \) using \( \phi = (1 + \tau) \phi_{+1} \) and dividing by \( \beta \) this yields

\[
\max_{\hat{m}} -i\phi_{+1} \hat{m} + \alpha (n) \{u(\hat{q}(\bar{m}))\} - \phi_{+1} \hat{m} \}
\]

which gives

\[
i\phi_{+1} = \alpha (n) \left\{\hat{q}' (\bar{m}) u'(\hat{q}(\bar{m}))\right\} - \phi_{+1} \}
\]

Using the implicit function theorem in (22) to plug \( \hat{q}' (\bar{m}) = \frac{\phi_{+1}}{g_I (\hat{q})} \) into the above equation one obtains

\[
\frac{u'(\hat{q})}{g_I'(\hat{q})} = 1 + \frac{i}{\alpha (n)}
\] (23)

which combined with

\[
c'(\hat{q}) = (1 - \delta)
\] (24)

characterize the quantity traded and produced, respectively.

**Type II [market production only]:** \( \hat{q} \geq \bar{q}. \)

We set up the Lagrangian

\[
L = \left[u (\hat{q}) - \phi_{+1} \hat{m}\right]^\theta \left[\phi_{+1} \hat{m} - (1 - \delta) \hat{q} - \{c(\hat{q}) - c(\bar{q})\}\right]^{1-\theta} + \lambda_1 (\bar{m} - \hat{m}).
\]

A similar exercise as above brings

\[
\phi_{+1} \hat{m} = g_{II} [\hat{q} (\bar{m})] = \frac{(1 - \theta) c'(\hat{q}) u(\hat{q}) + \theta u'(\hat{q}) [(1 - \delta) \hat{q} + c(\hat{q}) - c(\bar{q})]}{\theta u'(\hat{q}) + (1 - \theta)c'(\hat{q})}
\]

and \( \hat{q} \) given by

\[
\frac{u'(\hat{q})}{g_{II}'(\hat{q})} = 1 + \frac{i}{\alpha (n)}
\]
A1 Proof of Lemma 1

The Nash bargaining problem is:

$$\max_{\hat{q} \leq q, \hat{m} \leq m} B(\hat{q}, \hat{m}) = [u(\hat{q}) - \phi_{+1}\hat{m}]^\theta [\phi_{+1}\hat{m} - (1-\delta)\hat{q}]^{1-\theta}. \quad (25)$$

The Nash bargaining solution must satisfy individual rationality. Hence, without loss of generality, the domain of \((\hat{q}, \hat{m})\) is restricted to

$$\mathcal{A} = \{(\hat{q}, \hat{m}) \in [0, \hat{q}] \times [0, \hat{m}] | u(\hat{q}) - \phi_{+1}\hat{m} \geq 0, \phi_{+1}\hat{m} - (1-\delta)\hat{q} \geq 0\}.$$  

\(\mathcal{A}\) is non-empty as \((\hat{q}, \hat{m}) = (0,0) \in \mathcal{A}\). Furthermore, \(\mathcal{A}\) is compact and \(B(\hat{q}, \hat{m})\) is continuous. Therefore the maximization problem is well-defined. If either \(\hat{q} = 0\) or \(\hat{m} = 0\), then \((\hat{q}, \hat{m}) = (0,0)\) solves the problem uniquely. Assume now \(\hat{q} > 0\) and \(\hat{m} > 0\). We can always find some \((\hat{q}, \hat{m}) \in \mathcal{A}\) such that \(B(\hat{q}, \hat{m}) > 0\). Hence all points on the boundary of \(\mathcal{A}\) are ruled out as a solution. In the following, we then conduct our analysis in the interior of \(\mathcal{A}\), \(\mathcal{A}^o = \{(\hat{q}, \hat{m}) \in [0, \hat{q}] \times [0, \hat{m}] | u(\hat{q}) - \phi_{+1}\hat{m} > 0, \phi_{+1}\hat{m} - (1-\delta)\hat{q} > 0\}\). We divide the analysis into four steps, and Figures A1 and A2 illustrate the essential features in the analysis.

**Step 1.** We first analyze the first-order effect of \(\hat{q}\). Taking a derivative of \(B(\hat{q}, \hat{m})\) w.r.t. \(\hat{q}\), we have

\[
B_q(\hat{q}, \hat{m}) = \theta [u(\hat{q}) - \phi_{+1}\hat{m}]^{\theta-1} [\phi_{+1}\hat{m} - (1-\delta)\hat{q}]^{1-\theta} u'(\hat{q}) - (1-\theta)(1-\delta) [u(\hat{q}) - \phi_{+1}\hat{m}]^\theta [\phi_{+1}\hat{m} - (1-\delta)\hat{q}]^{-\theta}
\]

\[
= [u(\hat{q}) - \phi_{+1}\hat{m}]^{\theta-1} [\phi_{+1}\hat{m} - (1-\delta)\hat{q}]^{-\theta} \theta(\phi_{+1}\hat{m} - (1-\delta)\hat{q})u'(\hat{q}) - (1-\theta)(1-\delta)(u(\hat{q}) - \phi_{+1}\hat{m}).
\]

\(B_q(\hat{q}, \hat{m}) = 0\) iff

\[
\hat{m} = \frac{g(\hat{q})}{\phi_{+1}} = \frac{1-\theta}{\theta u'(\hat{q}) + (1-\theta)(1-\delta)} \frac{u(\hat{q})}{\phi_{+1}} + \frac{\theta u'(\hat{q})}{\theta u'(\hat{q}) + (1-\theta)(1-\delta)} \frac{1-\delta}{\phi_{+1}}.
\]

Let \((\bar{q}, \bar{m})\) be the unique positive solution of the system of equations \(u(\bar{q}) - \phi_{+1}\bar{m} = 0\) and \(\phi_{+1}\bar{m} - (1-\delta)\bar{q} = 0\). For all \((\hat{q}, \hat{m}) \in \mathcal{A}\), \((\hat{q}, \hat{m}) \leq (\bar{q}, \bar{m})\). It can be shown that \(g(\hat{q})\) is strictly
\begin{align*}
m &= \frac{(1-\delta)q}{\varphi+1} \\
m &= \frac{h(q)}{\varphi+1} \\
m &= \frac{g(q)}{\varphi+1} \\
m &= \frac{u(q)}{\varphi+1}
\end{align*}
increasing in \([0, \tilde{q}]\). Thus, the inverse of \(g(\cdot), g^{-1}(\cdot)\), exists in \([0, \tilde{q}]\). Accordingly, the first-order effect of \(\tilde{q}\) for any given \(\tilde{m} \in (0, g(\tilde{q}))\) can be summarized as:

\[
\text{sign } B_{\tilde{q}}(\tilde{q}, \tilde{m}) = \text{sign } (g^{-1}(\phi_{+1}\tilde{m}) - \tilde{q}).
\]

**Step 2.** Next we analyze the first-order effect of \(\tilde{m}\). Taking a derivative of \(B(\tilde{q}, \tilde{m})\) w.r.t. \(\tilde{m}\), we have

\[
B_{\tilde{m}}(\tilde{q}, \tilde{m}) = 0 \text{ iff } \tilde{m} = \frac{h(\tilde{q})}{\phi_{+1}} \equiv (1 - \theta) \frac{u(\tilde{q})}{\phi_{+1}} + \theta \frac{(1 - \delta) \tilde{q}}{\phi_{+1}}.
\]

Accordingly, the first-order effect of \(\tilde{m}\) for any given \(\tilde{q} \in (0, \tilde{q})\) can be summarized as:

\[
\text{sign } B_{\tilde{m}}(\tilde{q}, \tilde{m}) = \text{sign } \left(\frac{h(\tilde{q})}{\phi_{+1}} - \tilde{m}\right).
\]

**Step 3.** Consider the unconstrained Nash bargaining problem. \((\tilde{q}, \tilde{m}) \in \mathcal{A}^o\) maximizes \(B\) only if \(B_{\tilde{q}}(\tilde{q}, \tilde{m}) = B_{\tilde{m}}(\tilde{q}, \tilde{m}) = 0\), and the Hessian matrix of \(B\) is negative semidefinite in \(\mathcal{A}^o\). It can be shown that the Hessian matrix of \(B\) is negative semidefinite in \(\mathcal{A}^o\). It is also straightforward to show that \(B_{\tilde{q}}(\tilde{q}, \tilde{m}) = B_{\tilde{m}}(\tilde{q}, \tilde{m}) = 0\) if either \(u'(\tilde{q}) = 1 - \delta\) or \(\tilde{q} = (1 - \delta) \tilde{q}\). The solutions that solve \(u(\tilde{q}) = (1 - \delta) \tilde{q}\) are ruled out as a maximizer, as they are on the boundary of \(\mathcal{A}\). The only candidate is \((q_N, m_N) \in \mathcal{A}^o\), where \(u'(q_N) = 1 - \delta\) and \(m_N = h(q_N)/\phi_{+1} = g(q_N)/\phi_{+1}\). It can be verified that \((q_N, m_N)\) is the unique maximizer for the unconstrained Nash bargaining problem.

**Step 4.** Consider now the constrained Nash bargaining problem. Pick any \(\tilde{q} > 0\) and \(\tilde{m} > 0\). We observe that both \(g(\tilde{q})/\phi_{+1}\) and \(h(\tilde{q})/\phi_{+1}\) are convex combinations of \(u(\tilde{q})/\phi_{+1}\) and \((1 - \delta) \tilde{q}/\phi_{+1}\). \(g(\tilde{q}) = h(\tilde{q}) = u(\tilde{q}) = (1 - \delta) \tilde{q}\) in \([0, \tilde{q}]\) iff \(\tilde{q} = 0\) or \(\tilde{q} = \tilde{q}\). \(g(\tilde{q}) = h(\tilde{q}) = g(\tilde{q}) = h(\tilde{q})\) in \((0, \tilde{q}]\) iff
\[ \hat{q} = q_N. \text{ It is straightforward to show that } g(\hat{q}) < h(\hat{q}) \text{ when } \hat{q} \in (0, q_N), \text{ and } g(\hat{q}) > h(\hat{q}) \text{ when } \hat{q} \in (q_N, \bar{q}). \] Distinguish four cases:

**Case 1** \( (\hat{q}, \hat{m}) \in D_1 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 > q_N \text{ and } x_2 > m_N \}. \) Obviously \( (\hat{q}, \hat{m}) = (q_N, m_N) \) is the maximizer.

**Case 2** \( (\hat{q}, \hat{m}) \in D_2 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_2 \leq \min \{g(x_1)/\phi_{+1}, m_N\} \}. \) Based on Step 1 and Step 2, it can be shown that \( (\hat{q}, \hat{m}) = (g^{-1}(\phi_{+1}\hat{m}), \hat{m}) \) is the maximizer.

**Case 3** \( (\hat{q}, \hat{m}) \in D_3 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 \leq \min \{h^{-1}(x_2)/\phi_{+1}, q_N\} \}. \) Based on Step 1 and Step 2, it can be shown that \( (\hat{q}, \hat{m}) = (\hat{q}, h(\hat{q})/\phi_{+1}) \) is the maximizer.

**Case 4** \( (\hat{q}, \hat{m}) \in D_4 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 < q_N, x_2 < m_N, \text{ and } h(x_1) < x_2 < g(x_1) \}. \) Based on Step 1 and Step 2, it can be shown that \( (\hat{q}, \hat{m}) = (\hat{q}, \hat{m}) \) is the maximizer.

In sum, we have the following result:

\[ (\hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) \equiv \arg \max_{\bar{q} \leq \hat{q}, \bar{m} \leq \hat{m}} B(\bar{q}, \bar{m}) = (\min \{g^{-1}(\phi_{+1}\bar{m}), q_N, \bar{q}\}, \min \{h(\bar{q})/\phi_{+1}, m_N, \bar{m}\}) \].

**B. Proofs of Propositions 2-3**

Given \( \bar{m} \geq 0 \), the seller’s maximization problem is:

\[
\max_{\bar{q} \geq 0} \Phi(\bar{q}) = -c(\bar{q}) + \beta \left\{ \frac{n}{\alpha(n)} \left[ \phi_{+1}\bar{m}(\bar{q}, \bar{m}) + (1 - \delta) (\bar{q} - \hat{q}(\bar{q}, \bar{m})) \right] + (1 - \frac{n}{\alpha(n)}) (1 - \delta) \bar{q} \right\},
\]

and given \( \bar{q} \geq 0 \), the buyer’s maximization problem is:

\[
\max_{\bar{m} \geq 0} \Psi(\bar{m}) = -\phi\bar{m} + \beta \left\{ \alpha(n) \left\{ u[\bar{q}(\bar{q}, \bar{m})] + \phi_{+1} [\bar{m} - \bar{m}(\bar{q}, \bar{m})] \right\} + [1 - \alpha(n)]\phi_{+1}\bar{m} \right\}.
\]

First we have the following observation:

**Claim B1.** The seller will never pick any \( \bar{q} < \bar{q}_L \), where \( \bar{q}_L \equiv c^{-1}(\beta \left( 1 - \frac{n}{\alpha(n)} \right) (1 - \delta)) > 0. \)

**Proof.** Given any \( \bar{m} \geq 0, \bar{m}(\bar{q}, \bar{m}) \) and \( (\bar{q} - \hat{q}(\bar{q}, \bar{m})) \) are nondecreasing in \( \bar{q} \). Then for all \( \bar{q} \in [0, \bar{q}_L) \), \( \Phi(\bar{q}) \geq -c(\bar{q}) + \beta \left( 1 - \frac{n}{\alpha(n)} \right) (1 - \delta) > 0. \) Hence \( \bar{q} < \bar{q}_L \) will never be chosen by the seller.

\[ \blacksquare \]
Suppose \((q^*, \bar{m}^*)\) is a pure strategy Nash equilibrium. Then \((q^*, \bar{m}^*)\) has the following properties:

**Claim B2.** \((q^*, \bar{m}^*) \in D_2 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_2 \leq \min \{g(x_1) / \phi, m_N\}\}.

**Proof.** By Claim B1, \(q > 0\). It suffices to show that for any \(q > 0\), \(\bar{m} > \min \{g(q) / \phi, m_N\}\) yields the buyer strictly lower payoff then \(\min \{g(q), m_N\}\) does. Distinguish two cases: (i) \(q \geq q_N\). For any \(\bar{m} > \min \{g(q) / \phi, m_N\}\) = \(m_N\), \(q(\bar{m}, \bar{m}) = q_N = q(\bar{q}, m_N)\) and \(\bar{m}(\bar{q}, \bar{m}) = m_N = m(q, m_N)\). Therefore \(\Psi(m_N) - \Psi(\bar{m}) = -\phi(m_N - \bar{m}) + \beta \phi^1(m_N - \bar{m}) > 0\). (ii) \(0 < q < q_N\). For any \(\bar{m} > \min \{g(q) / \phi, m_N\}\) = \(g(\bar{q}) / \phi\), \(q(\bar{q}, \bar{m}) = q = q(\bar{q}, g(q) / \phi_{+1})\) and \(\bar{m}(\bar{q}, \bar{m}) = \min \{m, h(\bar{q}) / \phi\} > \bar{m}(\bar{q}, g(q) / \phi_{+1}) = g(\bar{q}) / \phi_{+1}\). Therefore \(\Psi(g(\bar{q}) / \phi_{+1}) - \Psi(\bar{m}) = (\beta \phi_{+1} - \phi)(g(q) / \phi_{+1} - \bar{m}) - \beta \alpha(n) \phi_{+1}(g(q) / \phi_{+1} - \bar{m}) > 0\). ■

**Claim B3.** \((q(\bar{q}^*, \bar{m}^*), \bar{m}(\bar{q}^*, \bar{m}^*)) = (g^{-1}(\phi_{+1} \bar{m}^*), \bar{m}^*)\).

**Proof.** From Claim B2, \((q^*, \bar{m}^*) \in D_2\), and \((q(\bar{q}), \bar{m}(\bar{q}, \bar{m})) = (g^{-1}(\phi_{+1} \bar{m}), \bar{m})\) for all \((\bar{q}, \bar{m})\) in \(D_2\) by Lemma 1. ■

**Claim B4.** \(q(\bar{q}^*, \bar{m}^*) = q^*\) only if \(q^* = q_N\).

**Proof.** Suppose to the contrary that \(q(q^*, \bar{m}^*) = q^*\) and \(q^* \neq q_N\). By Claim B3, \(g^{-1}(\phi_{+1} \bar{m}^*) = q^*\). Thus, \((q^*, \bar{m}^*) = (g^{-1}(\phi_{+1} \bar{m}^*), \bar{m}^*)\) and \(q^* \leq q_N\) as \((q^*, \bar{m}^*) \in D_2\). Since \(q^* \neq q_N\), \(q^* < q_N\).

As \(q^* = g^{-1}(\phi_{+1} \bar{m}^*)\) maximizes the seller’s payoff given \(\bar{m}^*\), the left hand derivative \(\Phi'\) at \(q^*\) is non-negative, and the right hand derivative at \(q^*\) is non-positive. The left hand derivative \(\Phi'\) at \(q^* = g^{-1}(\phi_{+1} \bar{m}^*)\) is

\[
\Phi'_{-}(q)|_{q = q^*} = \frac{d}{dq}[-c(q) + \beta\left(\frac{\alpha(n)}{n}\right)\left[\phi_{+1} \bar{m}(\bar{q}, \bar{m}^*) + (1 - \delta)(\bar{q} - q(q, \bar{m}^*))\right] + (1 - \frac{\alpha(n)}{n})(1 - \delta)\bar{q}]|_{q = q^*} = \frac{d}{dq}[-c(q) + \beta\left(\frac{\alpha(n)}{n}\right)\left[\phi_{+1} \bar{m}^* + (1 - \delta)(\bar{q} - q)\right] + (1 - \frac{\alpha(n)}{n})(1 - \delta)\bar{q}]|_{q = q^*} = -c'(q^*) + \beta(1 - \frac{\alpha(n)}{n})(1 - \delta),
\]

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and the right hand derivative $\Phi'_+)$ at $q^* = g^{-1}(\phi_{+1}\hat{m}^*)$ is

$$
\Phi'_+(q)|_{q=q^*_+} = \frac{d}{dq} [-c(q) + \beta \left( \alpha \frac{n}{n} [\phi_{+1}\hat{m}(q, \hat{m}^*) + (1 - \delta) (\hat{q} - \hat{q}(q, \hat{m}^*))] + (1 - \alpha \frac{n}{n}) (1 - \delta) \hat{q}] \right)_{q=q^*_+} \\
= \frac{d}{dq} [-c(q) + \beta \left( \alpha \frac{n}{n} [\phi_{+1}\hat{m}^* + (1 - \delta) (\hat{q} - g^{-1}(\phi_{+1}\hat{m}^*))] + (1 - \alpha \frac{n}{n}) (1 - \delta) \hat{q}] \right)_{q=q^*_+} \\
= -c'(q^*) + \beta (1 - \delta).
$$

Obviously $\Phi'_-(q)|_{q=q^*_-} \geq 0$ implies $\Phi'_+(q)|_{q=q^*_+} > 0$, a contradiction. Hence we must have $q^*_+ = q_N$.

(i) **Type I equilibrium.**

Now pick any $q \geq q_L > 0$, the first order effect of $\hat{m}$ at $\hat{m} = 0$ on $\Psi(\hat{m})$ is

$$
\Psi_{\hat{m}}(\hat{m})|_{\hat{m}=0} = \frac{d}{d\hat{m}} [-\phi\hat{m} + \beta \{ \alpha (n) \{ u(\hat{q}(q^*, \hat{m})) + \phi_{+1} [\hat{m} - \hat{m}(q^*, \hat{m})] \} + (1 - \alpha \frac{n}{n}) \phi_{+1}\hat{m} \}]|_{\hat{m}=0} \\
= \frac{d}{d\hat{m}} [-\phi\hat{m} + \beta \{ \alpha (n) u(g^{-1}(\phi_{+1}\hat{m}^*)) + (1 - \alpha \frac{n}{n}) \phi_{+1}\hat{m} \}]|_{\hat{m}=0} \\
= \beta (1 - \alpha \frac{n}{n}) \phi_{+1} - \phi + \beta \alpha \frac{n}{n} \frac{u'(\hat{q}(q^*, 0))}{g'(\hat{q}(q^*, 0))} \phi_{+1} \\
= \beta (1 - \alpha \frac{n}{n}) \phi_{+1} - \phi + \beta \alpha \frac{n}{n} \frac{u'(0)}{g'(0)} \phi_{+1},
$$

where

$$
g'(0) = \frac{d}{dq} \left[ \left( 1 - \delta \right) \left( 1 - \theta \right) u(q) \right]_{q=0} + \frac{\theta u'(q)}{\theta u'(q) + (1 - \theta) (1 - \delta)} \left( 1 - \delta \right) \hat{q} |_{q=0} \\
= \cdots \\
= \frac{(1 - \delta) u'(\hat{q})}{\theta u'(\hat{q}) + (1 - \theta) (1 - \delta)} + \frac{(1 - \delta) (1 - \theta) \theta u''(\hat{q}) [(1 - \delta) \hat{q} - u(q)]}{[\theta u'(\hat{q}) + (1 - \theta) (1 - \delta)]^2} |_{q=0} \\
= \frac{(1 - \delta) u'(0)}{\theta u'(0) + (1 - \theta) (1 - \delta)}.
$$

Hence $\Psi_{\hat{m}}(\hat{m})|_{\hat{m}=0} \leq 0$ if $\frac{u'(0)}{g'(0)} \leq 1 + \frac{i}{\alpha(n)}$, or equivalently, $u'(0) \leq (1 + \frac{i}{\alpha(n)}) (1 - \delta)$. Moreover, as $\frac{u'(x)}{g'(x)}$ is decreasing in $x$, it can be verified easily that given any $\bar{q} > 0$, $\Psi_{\hat{m}}(\hat{m}) < 0$. Thus, $\hat{m} = 0$ is the best response for the buyer. Given $\hat{m} = 0$, the seller will pick $\hat{q} = \hat{q}_H \equiv c^{-1} (\beta (1 - \delta))$.

Hence $(q^*, \hat{m}^*) = (\hat{q}_H, 0)$ is the unique Nash equilibrium.

(ii) **Type II Equilibrium.** First we show that there exists a unique pure strategy Nash equilibrium, then it is the unique Nash equilibrium. Let $(\bar{q}^*, \hat{m}^*)$ be a pure strategy Nash equilibrium. The condition (iii)(a) implies that $\hat{m}^* \notin \{0, m_N\}$. We claim that
\(\bar{q}(\bar{q}^*, \bar{m}^*) < \bar{q}^*\). Suppose to the contrary that \(\bar{q}(\bar{q}^*, \bar{m}^*) = \bar{q}^*\). By Claim B4, \(\bar{q}^* = q_N\). By Claim B3, \(g^{-1}(\phi_{+1}\bar{m}^*) = \bar{q}(\bar{q}^*, \bar{m}^*) = q_N\), which in turn implies \(\bar{m}^* = m_N\), contradicting the fact that \(\bar{m}^* \notin \{0, m_N\}\). Hence we must have \(\bar{q}(\bar{q}^*, \bar{m}^*) < \bar{q}^*\), which is guaranteed by condition (iii)(b).

**Claim B5.** \(\frac{u'(\bar{q}(\bar{q}^*, \bar{m}^*))}{g'(\bar{q}(\bar{q}^*, \bar{m}^*))} = \frac{u'(g^{-1}(\phi_{+1}\bar{m}^*))}{g'(g^{-1}(\phi_{+1}\bar{m}^*))} = \frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha(n)}\).

**Proof.** Since \(\bar{m}^* \notin \{0, m_N\}\) is an interior solution that solves the buyer’s maximization problem, the first-order effect vanishes. By Claims B3 and B4, \(\bar{q}(\bar{q}^*, \bar{m}^*) = g^{-1}(\phi_{+1}\bar{m}^*) < \bar{q}^*\) is not binding. We then have

\[
\Psi_{\bar{m}}(\bar{m})|_{\bar{m} = \bar{m}^*} = \frac{d}{d\bar{m}} \left[ -\phi\bar{m} + \beta \{ \alpha(n) \{ u(\bar{q}(\bar{q}^*, \bar{m})) + \phi_{+1}[\bar{m} - \bar{m}(\bar{q}^*, \bar{m})] \} + [1 - \alpha(n)\phi_{+1}\bar{m}] \} \right] |_{\bar{m} = \bar{m}^*} = \frac{d}{d\bar{m}} \left[ -\phi\bar{m} + \beta \{ \alpha(n) u(g^{-1}(\phi_{+1}\bar{m})) + [1 - \alpha(n)\phi_{+1}\bar{m}] \} \right] |_{\bar{m} = \bar{m}^*} = \beta(1 - \alpha(n))\phi_{+1} - \phi + \beta\alpha(n) \frac{u'(\bar{q}(\bar{q}^*, \bar{m}^*))}{g'(\bar{q}(\bar{q}^*, \bar{m}^*))} \phi_{+1} = 0.
\]

Hence

\[
\frac{u'(\bar{q}(\bar{q}^*, \bar{m}^*))}{g'(\bar{q}(\bar{q}^*, \bar{m}^*))} = \frac{u'(g^{-1}(\phi_{+1}\bar{m}^*))}{g'(g^{-1}(\phi_{+1}\bar{m}^*))} = \frac{u'((\zeta))}{g'(\zeta)} = 1 + \frac{\tau - \beta(1 - \alpha(n))}{\beta\alpha(n)} = 1 + \frac{i}{\alpha(n)}.
\]

**Claim B6.** \(\bar{q}^* = \bar{q}_H \equiv c^{-1}(\beta(1 - \delta)).\)

**Proof.** Since \(\bar{q}^* > \bar{q}(\bar{q}^*, \bar{m}^*)\) maximizes the seller’s problem, the first-order effect vanishes at \(\bar{q}^*\). Using Claim B3, we have

\[
\Phi_{\bar{q}}(\bar{q})|_{\bar{q} = \bar{q}^*} = \frac{d}{d\bar{q}} \left[ -c(\bar{q}) + \beta \left( \frac{\alpha(n)}{n} \left[ \phi_{+1}\bar{m}(\bar{q}, \bar{m}^*) + (1 - \delta)(\bar{q} - \bar{q}(\bar{q}, \bar{m}^*)) \right] + (1 - \frac{\alpha(n)}{n})(1 - \delta)\bar{q} \right] \right] |_{\bar{q} = \bar{q}^*} = \frac{d}{d\bar{q}} \left[ -c(\bar{q}) + \beta(1 - \delta)\bar{q} \right] |_{\bar{q} = \bar{q}^*} = -c'(\bar{q}^*) + \beta(1 - \delta) = 0.
\]

Finally, we need to check that \(\bar{q}_H\) indeed maximizes the seller’s payoff given \(\bar{m}^* = g(\zeta)/\phi_{+1}\).
The payoff at $\bar{q}_H$, which is the unique maximizer on $[\zeta, \infty)$, is

$$\Phi(\bar{q}_H) = -c(\bar{q}_H) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \phi_{+1}\bar{m}(\bar{q}_H, \bar{m}^*) + (1 - \delta)(\bar{q}_H - \bar{q}(\bar{q}_H, \bar{m}^*)) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q}_H \right\}$$

$$= -c(\bar{q}_H) + \beta \left\{ \frac{\alpha(n)}{n} \left[ g(\zeta) + (1 - \delta)(\bar{q}_H - \zeta) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q}_H \right\}.$$  

It is easy to verify that the seller will never pick any $\bar{q} < h^{-1}(g(\zeta)/\phi_{+1})$, and the payoff at any $\bar{q} \in [h^{-1}(g(\zeta)/\phi_{+1}), \zeta]$ is

$$\Phi(\bar{q}) = -c(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \phi_{+1}\bar{m}(\bar{q}, \bar{m}^*) + (1 - \delta)(\bar{q} - \bar{q}(\bar{q}, \bar{m}^*)) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q} \right\}$$

$$= -c(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{n} \left[ g(\zeta) + (1 - \delta)(\bar{q} - \zeta) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q} \right\}.$$  

The first order effect vanishes at $\bar{q} = \bar{q}_L$. If $\bar{q}_L < h^{-1}(g(\zeta)/\phi_{+1})$, then $h^{-1}(g(\zeta)/\phi_{+1})$ is the unique maximizer on $[h^{-1}(g(\zeta)/\phi_{+1}), \zeta]$. If $\bar{q}_L \in [h^{-1}(g(\zeta)/\phi_{+1}), \zeta]$, then $\bar{q}_L$ is the unique maximizer on $[h^{-1}(g(\zeta)/\phi_{+1}), \zeta]$. If $\bar{q}_L \geq \zeta$, then it can be readily seen that $\bar{q}_H$ is a global maximizer. Accordingly, $\bar{q}_H$ is a global maximizer iff

$$-c(\bar{q}_H) + \beta \left\{ \frac{\alpha(n)}{n} \left[ g(\zeta) + (1 - \delta)(\bar{q}_H - \zeta) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q}_H \right\}$$

$$\geq -c(\max\{h^{-1}(g(\zeta)/\phi_{+1}), \bar{q}_L\}) + \beta \left\{ \frac{\alpha(n)}{n} g(\zeta) + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \max\{h^{-1}(g(\zeta)/\phi_{+1}), \bar{q}_L\} \right\},$$

which holds under the condition $\zeta \leq g^{-1}(\phi_{+1}\bar{m}_C)$.

**C. Proof of Proposition 4**

First we show the following:

**Claim C1.** There exists no pure strategy Nash equilibrium when $\frac{u''(0)}{g'(0)} > 1 + \frac{i}{a(n)}$ and $\zeta > g^{-1}(\phi_{+1}\bar{m}_C)$.

**Proof.** Omitted. ■

As here the strategy domain $\mathcal{R}_+$ for both the buyer and seller is continuous, without causing any ambiguity, the mixed strategy must be defined properly before proceeding to the proof. A mixed strategy, $\mu : B_{\mathcal{R}_+} \rightarrow [0, 1]$, is a probability distribution (Borel probability measure) on $\mathcal{R}_+$, where $B_{\mathcal{R}_+}$ stands for the Borel $\sigma$–algebra in $\mathcal{R}_+$. $\mu$ is null on $A \in B_{\mathcal{R}_+}$ if $\mu(A) = 0$. $\mu$ is discrete if there exists a countable collection of points $\{a_i\} \subset \mathcal{R}_+$ such that $\sum \mu(a_i) = 1$. $\mu$ is
said to be *degenerate* on $A \in \mathbb{B}_{\mathbb{R}_+}$ if $\mu(a) > 0$ for some $a \in A$ and $\mu(C) = 0$ for all $C \in \mathbb{B}_{\mathbb{R}_+}$ such that $a \notin C$ and $C \subset A$. If $\mu$ is degenerate on $\mathbb{R}_+$, then $\mu(a) = 1$ for some $a \in \mathbb{R}_+$. Given a mixed strategy $\mu$, let $\mathcal{F}(x) = \mu([0, x])$ denote the distribution function of $\mu$. $\mathcal{F}$ is increasing and right continuous. Furthermore, $\mathcal{F}$ is differentiable a.e. Let $\{\omega_i\}$ with $\omega_i < \omega_{i+1}$ be the collection of points in $[0, \infty)$ at which $\mathcal{F}$ is not differentiable. Denote by $f(x) \equiv \mathcal{F}'(x)$ whenever $\mathcal{F}'(x)$ exists, and assume $\sup\{f(x) | x \in \mathbb{R}_+ \setminus \{\omega_i\}\} < \infty$. Then $\mathcal{F}$ is absolutely continuous on $[0, \infty) \setminus \{\omega_i\}$. By the Fundamental Theorem of Calculus for Lebesque Integral, $\mathcal{F}(b) - \mathcal{F}(a) = \int_a^b f(x) dx$ for any $[a, b] \subset (\omega_i, \omega_{i+1})$. Denote by $\int_{\omega_{i+1}}^{\omega_i} f(x) dx \equiv \lim_{y \downarrow \omega_i} \lim_{z \uparrow \omega_{i+1}} \int_y^z f(x) dx = \mathcal{F}(\omega_{i+1}) - \mathcal{F}(\omega_i)$. Denote by $\chi_A(x) : \mathbb{R}_+ \rightarrow \{0, 1\}$ the indicator function of $A \subset \mathbb{R}_+. \chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ otherwise.

Now we are ready to prove the Proposition. Let $(\mu_s, \mu_b)$ constitute a Nash equilibrium. Denote by $\mathcal{F}_b : \mathbb{R}_+ \rightarrow [0, 1]$ the distribution function induced by the buyer’s mixed strategy $\mu_b$, and $\mathcal{F}_s : \mathbb{R}_+ \rightarrow [0, 1]$ the distribution function induced by the seller’s mixed strategy $\mu_s$. Define $l(\mathcal{F}_i) = \sup\{x \in \mathbb{R}_+ : \mathcal{F}_i(x) = 0\}$ and $u(\mathcal{F}_i) = \inf\{x \in \mathbb{R}_+ : \mathcal{F}_i(x) = 1\}$, where $i \in \{b, s\}$. We prove the Proposition by establishing a sequence of claims:

**Claim C2.** $\bar{q}_L \leq l(\mathcal{F}_s) < u(\mathcal{F}_s) \leq \bar{q}_H$ and $g(\bar{q}_L)/\phi_{+1} \leq l(\mathcal{F}_b) \leq u(\mathcal{F}_b) \leq g(\bar{q}_H)/\phi_{+1}$.

**Proof.** First we observe that any $\bar{q} < \bar{q}_L$ will not be chosen by the seller with a positive probability, as it is strictly dominated by $\bar{q}_L$ (Claim B1). Hence $\bar{q}_L \leq l(\mathcal{F}_s)$. Second, we argue that $l(\mathcal{F}_s) < u(\mathcal{F}_s)$. Suppose not and $l(\mathcal{F}_s) = u(\mathcal{F}_s)$, i.e., $\mathcal{F}_s$ is degenerate. Then the strict monotonicity of $u'/g'$ implies that there is a unique $\bar{m}*$ that maximizes the buyer’s payoff; thus, $\mathcal{F}_b$ must be degenerate as well. But it contradicts Claim C1. Therefore $\mathcal{F}_s$ must be non-degenerate, i.e., $l(\mathcal{F}_s) < u(\mathcal{F}_s)$. Let $\bar{q}_U \equiv \left\{x \in \mathbb{R}_+: c'(x) = \beta \left[\frac{\alpha(n)}{n} h'(x) + \left(1 - \frac{\alpha(n)}{n}\right)(1 - \delta)\right]\right\}$. On the other hand, since $\bar{q}_U < q_N$, $\bar{q}_L < \bar{q}_H < q_N$. Given any $\bar{q} \geq \bar{q}_L$, $\frac{u'(q_N)}{g'(q_N)} \geq 1 + \frac{i}{\alpha(n)}$ implies that any $\bar{m} < g(\bar{q}_L)/\phi_{+1}$ will not be chosen by the buyer with a positive probability, as it is strictly dominated by $g(\bar{q}_L)/\phi_{+1}$. Hence $\bar{q}_L \leq l(\mathcal{F}_s)$ implies that $l(\mathcal{F}_b) \geq g(\bar{q}_L)/\phi_{+1}$.

Next we argue that $u(\mathcal{F}_s) \leq \bar{q}_H$. Given any $\bar{m} \geq 0$, any $\bar{q} > \bar{q}_U$ is strictly dominated by $\bar{q}_U$, as $\Phi_{\bar{q}}$ is always negative for all $\bar{q} > \bar{q}_U$. Therefore $u(\mathcal{F}_s) \leq \bar{q}_U$. Given $u(\mathcal{F}_s) \leq \bar{q}_U$, the buyer will not pick any $\bar{m} > g(\bar{q}_U)/\phi_{+1}$, as any such $\bar{m}$ is strictly dominated by $g(\bar{q}_U)/\phi_{+1}$. Consequently
Given any $u$ can find a finite sequence of the form in which the last term is no greater than $u$.

Proof. We show the following:

(1) It can be readily seen that any $\bar{q} > \bar{q}_H$ is strictly dominated by $\bar{q}_H$. Then the seller will never pick any $\bar{q} > \bar{q}_H$ with a positive probability and hence $u(F_s) \leq \bar{q}_H$. Given any $\bar{m} \leq g(\bar{q}_V)/\phi_{+1}$, any $\bar{q} > h^{-1}(g(\bar{q}_V)/\phi_{+1})$ is strictly dominated by $h^{-1}(g(\bar{q}_V)/\phi_{+1})$. Hence $u(F_0) \leq g(\bar{q}_V)/\phi_{+1}$ implies $u(F_s) \leq h^{-1}(g(\bar{q}_V)/\phi_{+1})$. Given $u(F_s) \leq h^{-1}(g(\bar{q}_V)/\phi_{+1})$, by the same token we have $u(F_0) \leq g(h^{-1}(g(\bar{q}_V)/\phi_{+1}))/\phi_{+1}$. Continuing in this fashion, we can find a finite sequence of the form \{$\bar{q}_U, h^{-1}(g(\bar{q}_V)/\phi_{+1}), h^{-1}(g(h^{-1}(g(\bar{q}_V)/\phi_{+1}))/\phi_{+1}), \ldots$\} in which the last term is no greater than $\bar{q}_H$. Applying the result in (i) gives us $u(F_s) \leq \bar{q}_H$.

$u(F_s) \leq \bar{q}_H$ directly implies that $u(F_0) \leq g(\bar{q}_H)/\phi_{+1}$.

In the following, we distinguish two cases:

**I.** $\bar{q}_L \geq h^{-1}(g(\bar{q}_H))$. By Claim C2, the support of $\mu_s \subset [\bar{q}_L, \bar{q}_H]$. Let \{$\omega_i$\} with $\omega_i < \omega_{i+1}$ be the collection of points in $[\bar{q}_L, \bar{q}_H]$ at which $F_s$ is not differentiable.

**Claim C3.** $\mu_b$ is degenerate, i.e., $\mu_b(\bar{m}^*) = 1$ for some $\bar{m}^* \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$.

**Proof.** We show the following:

(i) $\Psi_{\bar{m}}$ ($\Psi_{\bar{m}}^+$ when it is not differentiable) is strictly decreasing on $[g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$.

Pick any $j \in \mathbb{Z}$. The buyer’s payoff at $\bar{m} \in (g(\omega_j)/\phi_{+1}, g(\omega_{j+1})/\phi_{+1})$ is

$$\Psi(\bar{m}) = -\phi \bar{m} + \beta \left\{ \alpha(n) \left\{ \sum_{i=-\infty}^{j-1} f_{\omega_{i+1}} u(x) dF_s + \sum_{i=-\infty}^{j} u(\omega_i) [F_s(\omega_i) - F_s(\omega_i -)] + \int_{\omega_{j+1}} g^{-1}(\phi_{+1}\bar{m}) u(x) dF_s + [1 - F_s(g^{-1}(\phi_{+1}\bar{m})] u(g^{-1}(\phi_{+1}\bar{m})) \right\} + \frac{1}{1 - \alpha(n) \phi_{+1}} \right\},$$

hence

$$\Psi_{\bar{m}}(\bar{m}) = [-\phi + \beta (1 - \alpha(n)) \phi_{+1}] + \beta \alpha(n) \left\{ u(g^{-1}(\phi_{+1}\bar{m})) f_s(g^{-1}(\phi_{+1}\bar{m})) \frac{1}{g^{-1}(\phi_{+1}\bar{m})} \phi_{+1} - u(g^{-1}(\phi_{+1}\bar{m})) f_s(g^{-1}(\phi_{+1}\bar{m})) \frac{1}{g^{-1}(\phi_{+1}\bar{m})} \phi_{+1} + [1 - F_s(g^{-1}(\phi_{+1}\bar{m}))] \frac{u(g^{-1}(\phi_{+1}\bar{m}))}{g^{-1}(\phi_{+1}\bar{m})} \phi_{+1} \right\}$$

$$= [-\phi + \beta (1 - \alpha(n)) \phi_{+1}] + \beta \alpha(n) \left\{ \left[ 1 - F_s(g^{-1}(\phi_{+1}\bar{m})) \right] \frac{u(g^{-1}(\phi_{+1}\bar{m}))}{g^{-1}(\phi_{+1}\bar{m})} \phi_{+1} \right\}$$

$$= \phi_{+1} \beta \alpha(n) \left\{ [1 - F_s(g^{-1}(\phi_{+1}\bar{m}))] \frac{u(g^{-1}(\phi_{+1}\bar{m}))}{g^{-1}(\phi_{+1}\bar{m})} \phi_{+1} \right\}.$$

Since $F_s$ is non-decreasing and $u'|g'$ is strictly decreasing, $\Psi_{\bar{m}}$ is strictly decreasing on $(g(\omega_j)/\phi_{+1}, g(\omega_{j+1})/\phi_{+1})$ for every $j \in \mathbb{Z}$. Furthermore, $F_s$ is right continuous implies $\Psi_{\bar{m}}$ is
right continuous. The right derivative of $\Psi$ at $g(\omega_j)/\phi_{+1}$ is no greater than the left derivative of $\Psi$ at $g(\omega_j)/\phi_{+1}$:

$$
\Psi^+_m\left(\frac{g(\omega_j)}{\phi_{+1}}\right) = \phi_{+1} \phi \alpha(n) \left\{ \left[ 1 - F_s(\omega_j) \right] \frac{u' (\omega_j)}{g' (\omega_j)} - (1 + \frac{i}{\alpha(n)}) \right\} \\
\leq \phi_{+1} \phi \alpha(n) \left\{ \left[ 1 - F_s(\omega_j) \right] \frac{u' (\omega_j)}{g' (\omega_j)} - (1 + \frac{i}{\alpha(n)}) \right\} \\
= \lim_{m \to g(\omega_j)/\phi_{+1}} \phi_{+1} \phi \alpha(n) \left\{ \left[ 1 - F_s(g^{-1}(\phi_{+1} \tilde{m})) \right] \frac{u' (g^{-1}(\phi_{+1} \tilde{m}))}{g' (g^{-1}(\phi_{+1} \tilde{m}))} - (1 + \frac{i}{\alpha(n)}) \right\} \\
= \Psi^-_m\left(\frac{g(\omega_j)}{\phi_{+1}}\right).
$$

As this holds for any $j \in \mathbb{Z}$, $\Psi^+_m$ (when it is not differentiable) is strictly decreasing on $[g(\tilde{q}_L)/\phi_{+1}, g(\tilde{q}_H)/\phi_{+1}]$.

(ii) $\Psi(\tilde{m})$ is continuous. It suffices to check the continuity at $g(\omega_{j+1})/\phi_{+1}$ for each $j$. Pick any $j \in \mathbb{Z}$. Then

$$
\Psi(g(\omega_{j+1})/\phi_{+1}) = -\phi g(\omega_{j+1})/\phi_{+1} + \beta \left\{ \alpha(n) \left\{ \sum_{i=-\infty}^{\omega_{j+1}} u(x) dF_s + \sum_{i=-\infty}^{j+1} u(\omega_i)[F_s(\omega_i) - F_s(\omega_i)] \right\} +[1 - F_s(\omega_{j+1})]u(\omega_{j+1}) \right\} +[1 - \alpha(n)]\phi_{+1} g(\omega_{j+1})/\phi_{+1} \\
+\beta \alpha(n) \left\{ \sum_{i=-\infty}^{\omega_{j+1}} u(x) dF_s + \sum_{i=-\infty}^{j} u(\omega_i)[F_s(\omega_i) - F_s(\omega_i)] +[1 - F_s(\omega_{j+1})]u(\omega_{j+1}) \right\} \\
= \lim_{\tilde{m} \to g(\omega_{j+1})/\phi_{+1}} \Psi(\tilde{m});
$$

hence $\Psi(\tilde{m})$ is continuous.

Combining (i) and (ii), we show that given $\mu_s$, the optimal level of money holding is unique. Therefore $\mu_b$ is degenerate. ■

**Claim C4.** $\mu_s$ is binary. More specifically, $\mu_s(\tilde{q}_L) > 0, \mu_s(\tilde{q}_H) > 0$, and $\mu_s(\tilde{q}_L) + \mu_s(\tilde{q}_H) = 1$.

**Proof.** When $\tilde{m} = g(\tilde{q}_H)/\phi_{+1}$, $\Phi(\tilde{q}_H) < \Phi(\tilde{q}_L)$. On the other hand, when $\tilde{m} = g(\tilde{q}_L)/\phi_{+1}$, $\Phi(\tilde{q}_H) > \Phi(\tilde{q}_L)$. Let $\tilde{m} \in (g(\tilde{q}_L)/\phi_{+1}, g(\tilde{q}_H)/\phi_{+1})$ be such that the seller is indifferent between

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choosing $\bar{q}_L$ and $\bar{q}_H$ when $\bar{m} = \tilde{m}$, that is, $\tilde{m}$ solves

$$-c(\bar{q}_L) + \beta \left\{ \frac{\alpha(n)}{n} \phi_{+1}\tilde{m} + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q}_L \right\} $$

$$= -c(\bar{q}_H) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \phi_{+1}\tilde{m} + (1 - \delta) (\bar{q}_H - g^{-1}(\phi_{+1}\tilde{m})) \right] + \left( 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \bar{q}_H \right\};$$

consequently

$$\tilde{m} = \frac{1}{\phi_{+1}} g(\bar{q}_H) - \frac{1}{\phi_{+1}} \int_{\bar{q}_L}^{\bar{q}_H} \left[ c'(x) - \beta(1 - \frac{\alpha(n)}{n}) (1 - \delta) \right] dx \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}].$$

The seller’s best response on $[g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ can be summarized as follows:

$$\bar{q}(\tilde{m}) = \begin{cases} 
\bar{q}_H & \tilde{m} < \tilde{m} \\\n\bar{q}_L & \tilde{m} = \tilde{m} \\\n\bar{q}_H & \tilde{m} > \tilde{m} \n \end{cases}.$$

By Claim C3, in equilibrium $\mu_b(\tilde{m}^*) = 1$ for some $\tilde{m}^* \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$. As there exists no pure strategy equilibria, in equilibrium we must have $\tilde{m}^* = \tilde{m}$, and the seller randomizes between $\bar{q}_L$ and $\bar{q}_H$ with $\mu_s(\bar{q}_L) > 0$, $\mu_s(\bar{q}_H) > 0$, and $\mu_s(\bar{q}_L) + \mu_s(\bar{q}_H) = 1$.

Combining Claims C3 and C4, we can now construct the equilibrium. Based on Claim 4, the buyer’s objective function for any $\tilde{m} \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ can be written as:

$$\Psi(\tilde{m}) = \mu_s(\bar{q}_H) \left\{ -\phi \tilde{m} + \beta [\alpha(n) u(g^{-1}(\phi_{+1}\tilde{m})) + (1 - \alpha(n)) \phi_{+1}\tilde{m}] \right\}$$

$$+ \mu_s(\bar{q}_L) \left\{ -\phi \tilde{m} + \beta [\alpha(n) u(\bar{q}_L) + (1 - \alpha(n)) \phi_{+1}\tilde{m}] \right\}$$

$$= [\beta (1 - \alpha(n)) \phi_{+1} - \phi] \tilde{m} + \beta \alpha(n) [\mu_s(\bar{q}_H) u(g^{-1}(\phi_{+1}\tilde{m})) + \mu_s(\bar{q}_L) u(\bar{q}_L)];$$

therefore

$$\Psi_{\tilde{m}}(\tilde{m}) = \phi_{+1} \beta \alpha(n) \left\{ \mu_s(\bar{q}_H) \frac{u'(g^{-1}(\phi_{+1}\tilde{m}))}{g'(g^{-1}(\phi_{+1}\tilde{m}))} - (1 + \frac{i}{\alpha(n)}) \right\}.$$ 

As $\tilde{m}^* = \tilde{m}$ in equilibrium, we must have $\Psi_{\tilde{m}}(\tilde{m}) = 0$. Accordingly,

$$\mu_s(\bar{q}_H) = (1 + \frac{i}{\alpha(n)}) \frac{g'(g^{-1}(\phi_{+1}\tilde{m}))}{u'(g^{-1}(\phi_{+1}\tilde{m}))}.$$ 

To sum up, the pair $(\mu_s, \mu_b)$ constructed below constitutes a unique Nash equilibrium:

$$\mu_b(\tilde{m}) = \begin{cases} 
1 & \tilde{m} = \tilde{m} \\
0 & \tilde{m} \neq \tilde{m} \n \end{cases}$$

$$\mu_s(\bar{q}) = \begin{cases} 
(1 + \frac{i}{\alpha(n)}) \frac{g'(g^{-1}(\phi_{+1}\tilde{m}))}{u'(g^{-1}(\phi_{+1}\tilde{m}))} & \bar{q} = \bar{q}_H \\
1 - (1 + \frac{i}{\alpha(n)}) \frac{g'(g^{-1}(\phi_{+1}\tilde{m}))}{u'(g^{-1}(\phi_{+1}\tilde{m}))} & \bar{q} = \bar{q}_L \n \end{cases}$$
\( \hat{q}_L < h^{-1}(g(\hat{q}_H)) \).

**Claim C5.** \( \mu_b \) is either null or degenerate on \([g(\hat{q}_L)/\phi_{+1}, h(\hat{q}_L)/\phi_{+1}]\).

**Proof.** Analogously to the argument in Claim C3, we can show that there exists at most one maximizer for the buyer on \([g(\hat{q}_L)/\phi_{+1}, h(\hat{q}_L)/\phi_{+1}]\).

**Claim C6.** \( \mu_s \) is at most binary on \([\hat{q}_L, h^{-1}(g(\hat{q}_H))]\).

**Proof.** By Claim C5, \( \mu_b \) is either null or degenerate on \([g(\hat{q}_L)/\phi_{+1}, h(\hat{q}_L)/\phi_{+1}]\). Let \( \mu_b(\hat{m}) \geq 0 \) for some \( \hat{m} \in [g(\hat{q}_L)/\phi_{+1}, h(\hat{q}_L)/\phi_{+1}] \). Let \( \mu_b(\hat{m}) = 0 \) for all \( \hat{m} \in [g(\hat{q}_L)/\phi_{+1}, \hat{m}] \), and \( \mu_b(\hat{m}) = \mu_b(\hat{m}) \) for all \( \hat{m} \in [\hat{m}, h(\hat{q}_L)/\phi_{+1}] \). Let \( \{\tau_i\} \) with \( \tau_i < \tau_{i+1} \) be the collection of points in \([h(\hat{q}_L)/\phi_{+1}, g(\hat{q}_H)/\phi_{+1}]\) at which \( F_b \) is not differentiable. Distinguish two subcases:

**Subcase 1.** \( \hat{m} \geq g(h^{-1}(g(\hat{q}_H)))/\phi_{+1} \). Pick any \( j \in \mathbb{Z} \). The seller’s payoff at \( \bar{q} \in (h^{-1}(\phi_{+1}\tau_j), h^{-1}(\phi_{+1}\tau_{j+1})) \) is

\[
\Phi(\bar{q}) = -c(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{\phi_{+1}} \left[ \mu_b(\hat{m}) \hat{m} + \sum_{i=-\infty}^{j-1} \int_{\tau_i}^{\tau_{i+1}} xdF_b + \sum_{i=-\infty}^{j} \tau_i[\mathcal{F}_b(\tau_i) - \mathcal{F}_b(\tau_{i-1})] \right] + \int_{\tau_j}^{h(\hat{q})/\phi_{+1}} xdF_b + (1 - \mathcal{F}_b(h(\hat{q})/\phi_{+1}))h(\bar{q})/\phi_{+1} \right) \\
+ \left( 1 - \frac{\alpha(n)}{\phi_{+1}} \right) (1 - \delta) \bar{q} \right\},
\]

therefore

\[
\Phi(\bar{q}) = -c'(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{\phi_{+1}} \left[ h(\bar{q})/\phi_{+1} f_b(h(\hat{q})/\phi_{+1})h'(\bar{q})/\phi_{+1} + (1 - \mathcal{F}_b(h(\hat{q})/\phi_{+1}))h'(\bar{q})/\phi_{+1} \right] \right\} + \left( 1 - \frac{\alpha(n)}{\phi_{+1}} \right) (1 - \delta)
\]

Since \( \mathcal{F}_b \) is non-decreasing and \(-c' \) and \( h'(\bar{q}) \) are strictly decreasing, \( \Phi(\bar{q}) \) is strictly decreasing on \((h^{-1}(\phi_{+1}\tau_j), h^{-1}(\phi_{+1}\tau_{j+1}))\) for every \( j \in \mathbb{Z} \). It can be readily seen that there is a unique maximizer for \( \Phi(\bar{q}) \) on \([\hat{q}_L, h^{-1}(g(\hat{q}_H))]\).

**Subcase 2.** \( \hat{m} < g(h^{-1}(g(\hat{q}_H)))/\phi_{+1} \). Following the same line of argument as in the proof of Subcase 1, there is a unique maximizer for \( \Phi(\bar{q}) \) on \([\hat{q}_L, g^{-1}(\phi_{+1}\hat{m})]\). Now pick any \( \bar{q} > g^{-1}(\phi_{+1}\hat{m}) \) and \( \hat{q} \in (h^{-1}(\phi_{+1}\tau_j), h^{-1}(\phi_{+1}\tau_{j+1})) \) for some \( j \in \mathbb{Z} \). The seller’s payoff at \( \bar{q} \) is

\[
\Phi(\bar{q}) = -c(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{\phi_{+1}} \left[ \mu_b(\hat{m}) \hat{m} + \sum_{i=-\infty}^{j-1} \int_{\tau_i}^{\tau_{i+1}} xdF_b + \sum_{i=-\infty}^{j} \tau_i[\mathcal{F}_b(\tau_i) - \mathcal{F}_b(\tau_{i-1})] \right] \\
+ \int_{\tau_j}^{h(\hat{q})/\phi_{+1}} xdF_b + (1 - \mathcal{F}_b(h(\hat{q})/\phi_{+1}))h(\bar{q})/\phi_{+1} \\
+ \mu_b(\hat{m}) \frac{\alpha(n)}{\phi_{+1}} (1 - \delta) [\hat{q} - g^{-1}(\phi_{+1}\hat{m})] \right\} + \left( 1 - \frac{\alpha(n)}{\phi_{+1}} \right) (1 - \delta) \bar{q}
\]

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therefore
\[
\Phi_\bar{q}(\bar{g}) = -c'(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{n} (1 - F_b(h(\bar{q})/\phi_1)) h'(\bar{q}) + \left( \mu_b(m) \frac{\alpha(n)}{n} + 1 - \frac{\alpha(n)}{n} \right) (1 - \delta) \right\}.
\]

Similarly, it can be shown that there is a unique maximizer for \( \Phi(\bar{q}) \) on \([g^{-1}(\phi_1 m), h^{-1}(g(\bar{q} H))]\).

Hence there are at most two local maximizers on \([\bar{q}_L, h^{-1}(g(\bar{q} H))]\).

Combining these two subcases, we show that \( \mu_s \) is at most binary on \([\bar{q}_L, h^{-1}(g(\bar{q} H))]\). ■

**Claim C7.** \( \mu_b \) is either null or discrete (at most trinary) on \([h(\bar{q}_L)/\phi_1, g(\bar{q} H)/\phi_1]\).

**Proof.** Thanks to Claim C6, let \( \bar{q}_1, \bar{q}_2 \in [\bar{q}_L, h^{-1}(g(\bar{q} H))] \) be such that \( \mu_s(\bar{q}_1) \geq 0, \mu_s(\bar{q}_2) \geq 0 \). Without loss of generality, let \( \bar{q}_1 < \bar{q}_1 < \bar{q}_2 < h^{-1}(g(\bar{q} H)) \). The cases for \( \bar{q}_L = \bar{q}_1 \) or \( \bar{q}_2 = h^{-1}(g(\bar{q} H)) \) can be analyzed analogously. Also let \( \{ \omega_i \} \) with \( \omega_i < \omega_i+1 \) be the collection of points in \([h^{-1}(g(\bar{q} H)), \bar{q}_H]\) at which \( F_s \) is not differentiable. First pick any \( \bar{m} \in (h(\bar{q}_L)/\phi_1, h(\bar{q}_1)/\phi_1) \) such that \( g^{-1}(\phi_1 \bar{m}) \neq \omega_i \) for all \( i \). \( g^{-1}(\phi_1 \bar{m}) \in (\omega_j, \omega_{j+1}) \) for some \( j \). The buyer’s payoff at \( \bar{m} \) is

\[
\Psi(\bar{m}) = -\phi \bar{m} + \beta \left\{ \right. \\
\left. \text{where } \right. \\
\sum_{i=1}^{2} \mu_s(\bar{q}_i) u(\bar{q}_i) + \sum_{i=-\infty}^{j-1} f_{\omega_{i+1}^{-}} u(x) dF_s \\
+ \sum_{i=-\infty}^{j} u(\omega_i) [F_s(\omega_i) - F_s(\omega_i^-)] \\
+ \int_{\omega_{j+1}}^{g^{-1}(\phi_1 \bar{m})} u(x) dF_s + [1 - F_s(g^{-1}(\phi_1 \bar{m}))] u(g^{-1}(\phi_1 \bar{m})) \\
\left. \right\}.
\]

It is straightforward to verify that \( \Psi(\bar{m}) \) is strictly decreasing on \((h(\bar{q}_L)/\phi_1, h(\bar{q}_1)/\phi_1) \) and \( \Psi(\bar{m}) \) is continuous; hence there is a unique local maximizer on \([h(\bar{q}_L)/\phi_1, h(\bar{q}_1)/\phi_1]\). Then \( \mu_b \) is either null or degenerate on \([h(\bar{q}_L)/\phi_1, h(\bar{q}_1)/\phi_1]\). Similarly, we can show that \( \mu_b \) is either null or degenerate on \([h(\bar{q}_1)/\phi_1, h(\bar{q}_2)/\phi_1]\) and \([h(\bar{q}_2)/\phi_1, g(\bar{q} H)/\phi_1]\) ■

**Claim C8.** \( \mu_s \) is either null or discrete (at most quinary) on \([h^{-1}(g(\bar{q} H)), \bar{q}_H]\).

**Proof.** By Claim C5, \( \mu_b \) is either null or degenerate on \([g(\bar{q} L)/\phi_1, h(\bar{q} L)/\phi_1]\). By Claim C7, \( \mu_b \) is at most trinary on \([h(\bar{q}_L)/\phi_1, g(\bar{q} H)/\phi_1]\). Let \( \bar{m}_1 \in [g(\bar{q}_L)/\phi_1, h(\bar{q}_L)/\phi_1] \) and \( \bar{m}_2, \bar{m}_3, \bar{m}_4 \in [h(\bar{q}_L)/\phi_1, g(\bar{q} H)/\phi_1] \) be such that \( \sum_{i=1}^{4} \mu_b(\bar{m}_i) = 1 \). Without loss of generality, let \( g(\bar{q}_L)/\phi_1 < \bar{m}_1 < \bar{m}_2 < \bar{m}_3 < \bar{m}_4 < g(\bar{q} H)/\phi_1 \). Denote by \( A_i \equiv \{ \bar{q} \in [h^{-1}(g(\bar{q} H)), \bar{q}_H] | \bar{q} \geq g^{-1}(\phi_1 \bar{m}_i) \} \). Consider two subcases:

\footnote{Note that \( \lim_{\bar{m} \to -h(\bar{q}_1)/\phi_1} \Psi(\bar{m}) < \Psi(\bar{m}(h(\bar{q}_1)/\phi_1)) \) if \( \mu_s(\bar{q}_1) > 0 \). Hence there could be multiple global maximizers.}
Subcase 1. $m_1 > g(h^{-1}(g(q_H))/\phi_+1)$. The seller’s payoff at $q \in [h^{-1}(g(q_H)), q_H]$ is

$$
\Phi(q) = -c(q) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \phi_+ \sum_{i=1}^{4} \mu_b(m_i) m_i + (1 - \delta) \sum_{i=1}^{4} \mu_b(m_i) \chi_{C_i}(q) |q - g^{-1}(\phi_+ m_i)| \right] \right\};
$$

hence $\forall q \in (h^{-1}(g(q_H)), q_H) \setminus \{g^{-1}(\phi_+ m_i)\}_{i=1}^{4}$,

$$
\Phi q(q) = -c'(q) + \beta \left\{ \frac{\alpha(n)}{n} \sum_{i=1}^{4} \mu_b(m_i) \chi_{C_i}(q) (1 - \delta) + \left(1 - \frac{\alpha(n)}{n}\right) (1 - \delta) \right\}.
$$

The collection $\{m_i\}$ divides $[h^{-1}(g(q_H)), q_H]$ into five intervals through $g^{-1}$. It can be readily seen that there exists a unique local maximizer on each interval. Therefore $\mu_s$ is either null or discrete (at most quinary) on $[h^{-1}(g(q_H)), q_H]$.

Subcase 2. $m_1 \leq g(h^{-1}(g(q_H))/\phi_+1)$. Similarly, it can be shown that $\mu_s$ is either null or discrete (at most quaternary) on $[h^{-1}(g(q_H)), q_H]$. $\blacksquare$

Claim C9. $\mu_b$ is at most trinary on $[g(q_L)/\phi_+1, g(q_H)/\phi_+1]$.

Proof. By Claims C6 and C8, $\mu_s$ is at most binary on $[q_L, h^{-1}(g(q_H))]$ and at most quinary on $[h^{-1}(g(q_H)), q_H]$. Let $\{q_i\}_{i=1}^{7}$ with $q_i < q_i+1$ be such that $\{q_1, q_2\} \subset [q_L, h^{-1}(g(q_H))]$, $\{q_i\}_{i=3}^{7} \subset [h^{-1}(g(q_H)), q_H]$, and $\sum_{i=1}^{7} \mu_s(q_i) = 1$. Denote by $B_i \equiv \{m \in [g(q_L)/\phi_+1, g(q_H)/\phi_+1]|m \geq g(q_i)/\phi_+1\}$, and $C_i \equiv \{m \in [g(q_L)/\phi_+1, g(q_H)/\phi_+1]|m \geq h(q_i)/\phi_+1\}$. The buyer’s payoff at $m \in [g(q_L)/\phi_+1, g(q_H)/\phi_+1]$ is

$$
\Psi(m) = -\phi m + \beta \left\{ \frac{\alpha(n)}{n} \left[ \sum_{i=1}^{7} \mu_s(q_i) \chi_{B_i}(m) u(q_i) + (1 - \chi_{B_i}(m)) u(g^{-1}(\phi_+ m)) \right] \right\};
$$

hence $\forall m \in (g(q_L)/\phi_+1, g(q_H)/\phi_+1) \setminus \{g(q_i)/\phi_+1\}_{i=1}^{7} \cup \{h(q_i)/\phi_+1\}_{i=1}^{2}$,

$$
\Psi_m(m) = \left[\phi + \beta (1 - \alpha(n)) \phi_+1 \right] + \beta \alpha(n) \left[ \sum_{i=1}^{7} \mu_s(q_i) (1 - \chi_{B_i}(m)) \frac{u'(g^{-1}(\phi_+ m))}{g^{-1}(\phi_+ m)} \phi_+1 \right].
$$

It can be readily verified that there exists a unique local maximizer on $[g(q_L)/\phi_+1, h(q_1)/\phi_+1]$, $[h(q_1)/\phi_+1, h(q_2)/\phi_+1]$, and $[h(q_2)/\phi_+1, g(q_H)/\phi_+1]$. Thus, $\mu_b$ is at most trinary on $[g(q_L)/\phi_+1, g(q_H)/\phi_+1]$.

Claim C10. $\mu_s$ is at most quaternary on $[q_L, q_H]$.
Proof. Thanks to Claim C9, let \( \{ \bar{m}_i \}^3_{i=1} \subset [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}] \) be such that \( \sum_{i=1}^3 \mu_b(\bar{m}_i) = 1 \). Denote by \( A_i \equiv \{ \bar{q} \in [\bar{q}_L, \bar{q}_H] \mid \bar{q} \geq g^{-1}(\phi_{+1}\bar{m}_i) \} \), and \( B_i \equiv \{ \bar{q} \in [\bar{q}_L, \bar{q}_H] \mid \bar{q} \leq h^{-1}(\phi_{+1}\bar{m}_i) \} \).

The seller’s payoff at \( \bar{q} \in [\bar{q}_L, \bar{q}_H] \) is

\[
\Phi(\bar{q}) = -c(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \phi_{+1} \sum_{i=1}^3 \mu_b(\bar{m}_i) \left[ \chi_{B_i}(\bar{q}) h(\bar{q})/\phi_{+1} + (1 - \chi_{B_i}(\bar{q})) \bar{m}_i \right] \right] + (1 - \alpha(n)) \left( 1 - \frac{\alpha(n)}{n} \right) \right\};
\]

hence \( \forall \bar{q} \in (\bar{q}_L, \bar{q}_H) \setminus \{ g^{-1}(\phi_{+1}\bar{m}_i) \}_{i=1}^3 \cup \{ h^{-1}(\phi_{+1}\bar{m}_i) \}_{i=1}^3 \),

\[
\Phi_{\bar{q}}(\bar{q}) = -c'(\bar{q}) + \beta \left\{ \frac{\alpha(n)}{n} \left[ \sum_{i=1}^3 \mu_b(\bar{m}_i) \left[ \chi_{B_i}(\bar{q}) h'(\bar{q}) + \chi_{A_i}(\bar{q}) (1 - \delta) \right] \right] + (1 - \alpha(n)) (1 - \delta) \right\}.
\]

The collection \( \{ \bar{m}_i \} \) divides \([\bar{q}_L, \bar{q}_H]\) into four intervals through \( g^{-1} \). It can be verified that there exists a unique local maximizer on each interval. Therefore \( \mu_s \) is at most quaternary on \([\bar{q}_L, \bar{q}_H] \).

Claim C11. \( \mu_b(g(\bar{q}_H)/\phi_{+1}) = 0 \).

Claim C12. \( \mu_s(\bar{q}_H) > 0 \).

D. Sufficient Conditions for Strictly Decreasing \( u'(x)/g'(x) \) on \([0, q_N]\).

Condition 1 \( u'(x)/g'(x) \) is strictly decreasing on \([0, q_N]\) if one of the following conditions is satisfied:

(i) \( \theta \) is not too small.

(ii) \( u''(x) \leq \frac{2\theta u'(x)^2}{\theta u'(x) + (1 - \theta)(1 - \delta)} \) on \([0, q_N]\).

(iii) \( u'(x) \) is log concave on \([0, q_N]\) and \( \theta \geq \frac{1}{2} \).

Proof. First we observe that \( (u'(x)/g'(x))' = u'(x)g'(x)[u''(x)/u'(x) - g''(x)/g'(x)] \). As \( u'(x) > 0 \) and for all \( x \in [0, q_N] \),

\[
g'(x) = \frac{(1 - \delta) u'(x)}{\theta u'(x) + (1 - \theta)(1 - \delta)} + \frac{(1 - \delta) (1 - \theta) \theta u''(x) [(1 - \delta) x - u(x)]}{\theta u'(x) + (1 - \theta)(1 - \delta)^2} > 0;
\]

\( (u'(x)/g'(x))' < 0 \) if and only if \( u''(x)/u'(x) - g''(x)/g'(x) < 0 \). Denote \( g'(x) = A(x) + B(x), \)

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where

\[ A(x) = \frac{(1 - \delta) u'(x)}{\theta u'(x) + (1 - \theta)(1 - \delta)} \text{ and } B(x) = \frac{(1 - \delta)(1 - \theta) \theta u''(x)\left[(1 - \delta)x - u(x)\right]}{[\theta u'(x) + (1 - \theta)(1 - \delta)]^2}; \]

then \( u''(x)/u'(x) - g''(x)/g'(x) = [u''(x)/u'(x) - A'(x)/g'(x)] - B'(x)/g'(x) \). As \( (1 - \delta)x - u(x) \leq 0 \) for all \( x \in [0, q_N] \), we have

\[
\frac{A'(x)}{g'(x)} = \frac{(1 - \delta)^2(1 - \theta)u''(x)}{\theta u'(x) + (1 - \theta)(1 - \delta)} + \frac{1 - (1 - \delta)(1 - \theta)u''(x)[(1 - \delta)x - u(x)]}{[\theta u'(x) + (1 - \theta)(1 - \delta)]^2} \geq \frac{(1 - \delta)(1 - \theta) u''(x)}{\theta u'(x) + (1 - \theta)(1 - \delta) u'(x)}.
\]

Accordingly we have

\[
\frac{u''(x)}{u'(x)} - \frac{A'(x)}{g'(x)} \leq \frac{u''(x)}{u'(x)}\left[1 - \frac{(1 - \delta)(1 - \theta)}{\theta u'(x) + (1 - \theta)(1 - \delta)}\right] \leq \frac{\theta}{\theta u'(x) + (1 - \theta)(1 - \delta)} u''(x) < 0.
\]

On the other hand, we have

\[
B'(x) = \frac{(1 - \delta)(1 - \theta) \theta}{[\theta u'(x) + (1 - \theta)(1 - \delta)]^2} \times [u''(x)((1 - \delta)x - u(x)) + u''(x)(1 - \delta - u'(x)) - \frac{2\theta(u''(x))^2((1 - \delta)x - u(x))}{\theta u'(x) + (1 - \theta)(1 - \delta)}].
\]

\( B'(x) \) vanishes when \( \theta \approx 1 \). Hence \( u''(x)/u'(x) - g''(x)/g'(x) < 0 \) when \( \theta \approx 1 \). This proves part (i).

For part (ii), first we observe that \( u''(x)(1 - \delta - u'(x)) \geq 0 \) for all \( x \in [0, q_N] \). Consequently \( B'(x) \geq 0 \) if

\[
u''(x)((1 - \delta)x - u(x)) \geq \frac{2\theta(u''(x))^2((1 - \delta)x - u(x))}{\theta u'(x) + (1 - \theta)(1 - \delta)},
\]

\[
\implies u''(x) \leq \frac{2\theta(u''(x))^2}{\theta u'(x) + (1 - \theta)(1 - \delta)};
\]

hence part (ii) is established.
To establish part (iii), let us assume \( u'(x) \) is log concave on \([0, q_N]\); hence \( u'''(x) \leq \frac{(u''(x))^2}{u'(x)} \).

Then \( B'(x) \geq 0 \) if

\[
\frac{(u''(x))^2}{u'(x)} \leq \frac{2\theta (u''(x))^2}{\theta u'(x) + (1 - \theta)(1 - \delta)} \iff u'(x) \geq \frac{1 - \theta}{\theta} (1 - \delta).
\]

The inequality holds if \( \theta \geq 1/2 \). Accordingly \( u''(x)/u'(x) - g''(x)/g'(x) < 0 \) if \( u'(x) \) is log concave on \([0, q_N]\) and \( \theta \geq \frac{1}{2} \). This proves part (iii).
References


