Interaction-based Foundation of Aggregate Investment Fluctuations

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Abstract

This paper demonstrates that the interactions of firm-level indivisible investments give rise to aggregate fluctuations without aggregate exogenous shocks. I develop a method to derive the distribution of the aggregate capital growth rate by embedding a fictitious tatonnement in a branching process. This method shows that idiosyncratic shocks may lead to non-vanishing aggregate fluctuations when the number of firms tends to infinity. By incorporating this mechanism in a dynamic general equilibrium model with indivisible investment and predetermined price, I provide the business cycle theory with a driver of fluctuations: aggregate investment demand fluctuations that arise from idiosyncratic productivity shocks. Due to predetermined prices of goods, firms respond to investment fluctuations by adjusting labor and output, thereby causing the comovements of output and consumption with investment. Numerical simulations show that the model generates aggregate fluctuations comparable to the business cycles in magnitude and correlation structure under standard calibration.

Keyword: Business cycle, strategic complementarity, idiosyncratic shock, law of large numbers, criticality, power law.

JEL Classification: E22, E32

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1 Introduction

One of the most significant questions in macroeconomics is what drives the short-run fluctuations of output in normal times. On the one hand, pursuing this question, researchers have investigated a number of shocks in such fundamental parameters as technology, preference, monetary policy, and expectations. While these aggregate exogenous shocks have been incorporated in the business cycles models and researchers have proven their empirical relevance, there remains a gap between the observed shocks and model interpretation. On the other hand, traditional Keynesians and practitioners in business and policy have stressed the role of autonomous shifts in aggregate investment demand. Nevertheless, the modern business cycles literature has stayed away from the investment demand shocks, since investment demand is an endogenous variable. This paper seeks to augment the literature by bringing these two lines of thought together. It gives a theoretical foundation to the use of investment demand shocks in the standard business cycles model by introducing interactions of firm-level non-linear investment decisions.

This paper shows that the interactions of indivisible investments can generate aggregate demand fluctuations. An indivisible investment is the simplest example of non-linearity observed in the real economy. When such non-linear behaviors are coupled with each other, the system can generate aggregate fluctuations. However, it is important to note that it only occurs in certain restricted environments. For example, if a pair-wise correlation of non-linear oscillations is too weak, the law of large numbers takes effect and suppresses fluctuations in aggregation. The right amount of correlation is necessary to generate aggregate fluctuations.

To illustrate this point, let us consider a simple reduced model. Suppose that there are \( N \) firms, each of which potentially conducts a binary investment. All firms are connected with each other and a firm’s investment causes the investment of each one of the other firms with probability \( \phi/N \). In this case, the mean number of firms induced to invest by a triggering investment is \( \phi \). If a triggering firm induces another firm to invest, the second investment induces another investment in turn. The mean and variance of the total number of firms induced to invest by this process are finite if \( \phi < 1 \) when \( N \) tends to infinity. Thus, the variance of the ratio of investing firms to \( N \) scales as \( 1/N^2 \) and quickly becomes negligible. If \( \phi > 1 \), the chain-reaction process is explosive, and thus, there is a non-negligible chance of all firms investing simultaneously, which does not agree with the observations of business cycles.
in normal times.

An interesting phenomenon emerges when $\phi = 1$. At this value, the above process stops in a finite step with probability one even in the infinite limit of $N$, whereas the mean and variance of the total number of investing firms diverge. When the number of initial triggering firms follows a binomial distribution with probability $\phi$ and population $N$, it turns out that the ratio of investing firms to $N$ has a non-zero asymptotic variance. This provides a foundation for aggregate investment demand fluctuations. At the critical value of connectedness $\phi$, micro investment decisions have macro consequences. Then, the central question is why $\phi$ has to have the particular critical value 1 in the real economy. This occurs when the investment threshold for each firm is proportional to aggregate capital. This paper presents a dynamic general equilibrium model in which such a proportional rule emerges naturally.

I consider monopolistic firms competing by producing differentiated goods that are consumed by households with homothetic preference. This economy features aggregate demand externality as in Blanchard and Kiyotaki (1987), with which an increase in aggregate demand proportionally shifts the demand schedule for each good. Under constant returns to scale technology, capital level would be indeterminate in the production sector if capital is adjusted continuously. Now suppose that a capital adjustment is a discrete decision. By incorporating indivisible investments, I obtain two results that do not arise in the case of continuous investments: the capital level given factor prices is locally unique, and the distribution of aggregate investment fluctuations is analytically derived.

Idiosyncratic shocks give rise to aggregate risks under these three conditions. First, the investment decision is non-linear. If the investment response to aggregate capital is smooth and locally linear, then the idiosyncrasies of micro-level investments cancel out with each other as the law of large numbers predicts. Second, the investment decisions complement each other. That is to say, even though the demand for toothbrush is discrete, it does not generate aggregate fluctuations as long as the household toothbrush demands are independent of each other. Third, the complementarity is large. It must be large enough for a firm’s investment to induce, on average, one other investment. This implies that the complementarity leads to indeterminacy if the investments are continuous rather than discrete. Considering these restrictive conditions helps us to identify possible loci where interactions pose aggregate risks. There are a few such aggregate phenomena in an economy:
one notable example is Keynes’ beauty contest of security traders; another is the pricing of a good given an aggregate price level. This paper proposes firm-level investment decisions as another phenomenon that meets these conditions.

This paper delivers three results. First, under the assumption of threshold investment rule, an asymptotic distribution function of the aggregate capital fluctuation is derived when the number of firms tends to infinity. The distribution has a heavier tail than the normal distribution. The fat tail indicates that the size of aggregate investment is sensitive to the detailed configuration of firms’ positions in the inaction band. This sensitivity to the detailed configuration causes the aggregate investment to exhibit fluctuations in the course of the evolution of capital profile driven by depreciation and discrete investments. Second, I show that the variance of aggregate fluctuations does not vanish at the infinite limit of the number of firms. Even though an economy consists of an infinite number of firms, the non-linear behavior at the firm level does not cancel out in aggregation. This result contrasts to the sectoral models that lack a strong amplification mechanism of idiosyncratic shocks due to the law of large numbers. Third, I numerically show that the dynamic general equilibrium with indivisible capital, predetermined price setting, and many but a finite number of firms generates aggregate fluctuations comparable to the business cycles in their magnitude and correlation structure.

I employ a fictitious tatonnement process to derive the distribution of aggregate fluctuation. An investment by a firm increases the aggregate capital and output in the next period. Because of the aggregate demand externality, the higher output induces the other firms to produce more in the next period and thus to invest more in this period. Then, there is a chance of a chain reaction of investments in which one firm’s investment triggers another. I formalize this chain reaction as a fictitious best-response dynamics that converges to an equilibrium. The size of the chain reaction depends on the configuration of firms’ positions in the inaction band. With a one-sided (S,s) policy, a firm’s position in the inaction band asymptotically follows a uniform distribution. Thus, I derive the unconditional distribution of aggregate investment size by drawing a profile of firms’ positions from a jointly uniform distribution with its dimension being equal to the number of firms.

Several studies have pointed out the synchronized timing of firms’ discrete actions as an important source of macroeconomic fluctuations. Shleifer (1986) demonstrated that the event of synchronized actions can recur deter-
ministically and endogenously through self-fulfilling expectations of periodic adjustments. Jovanovic (1987) posed the question as to when idiosyncratic shocks give rise to aggregate risks. Durlauf (1993) showed that the aggregate size of synchronized actions depends on the detailed configuration of agents’ states and can exhibit a long-run path-dependence. I extend this literature by presenting a sharper characterization of the synchronization in a standard business cycle model. I obtain an analytical expression of the fluctuation magnitude with parameters that can be estimated from firm-level data.

Scholars working on interaction-based models and those working on (S,s) economies, independently from each other, have tackled the question of how to analyze the aggregate fluctuations that arise from micro-level discreteness, or more generally, micro-level non-linearity. The models of interactions and non-linear dynamics have shed light on the possibility of endogenous fluctuations arising from micro-level non-linearity, as in Glaeser, Sacerdote, and Scheinkman (1996), Brock and Hommes (1997), and Brock and Durlauf (2001). The (S,s) literature, in contrast, concentrates on macroeconomy where pricing or investment incurs fixed costs and thus exhibits non-linearity at the micro level. Typically, an aggregate (S,s) model features a continuum of firms as in Thomas (2002). This modeling choice precludes the possibility in which interactions of “granular” firms give rise to aggregate fluctuations as in the interaction-based models. While I draw on the (S,s) literature in some important respects, the fluctuation results of this paper are obtained in the model with many but a finite number of firms, and the intuition of the results is analogous to that of the interaction-based models.

This paper contributes to the ongoing debate on the origins of business cycle fluctuations. Researchers such as Fisher (2006) and Justiniano, Primiceri, and Tambalotti (2010) have shown the importance of investment-specific technology shocks in accounting for business cycles in dynamic general equilibrium models. This paper provides a microfoundation for the autonomous fluctuations of aggregate investments. This paper shares its motivation with the literature on sunspot equilibrium, such as Galí (1994) and Wang and Wen (2008), but it differs in that the agents' expectation system is dynamically determinate. In this model, the agents' expectation system is a continuum approximation of the equilibrium dynamics which is a system of non-linear dynamics of a finite number of firms. Unlike sunspot models, the equilibrium outcome is locally unique due to the discreteness of micro-level decisions. The mechanism for scale-free fluctuations in this model is related to
the “break of the law of large numbers” argument in Gabaix (2011) and Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012). In their approach, the break is caused by fat-tailed distributions of firm size or the firms’ heterogeneous influence on other firms. This paper complements their findings by demonstrating the aggregate fluctuations even when the firm size and the influence vector are homogeneous. The fluctuation mechanism in this paper is most closely related to the self-organized criticality models (Bak, Chen, Scheinkman, and Woodford (1993)). In those models, an individual action causes an “avalanche” of other actions, and the avalanche size follows a fat-tail distribution. While the preceding self-organized criticality models feature locally interacting firms, this paper is concerned with firms that interact globally (i.e., with all the other firms) in goods markets in a dynamic general equilibrium.

The rest of this paper is organized as follows. Section 2 analytically characterizes the aggregate fluctuations that arise from threshold behaviors and complementarity without aggregate shocks. I obtain the distribution function of the fluctuations under a behavioral investment rule and random initial capital, which will be given a micro-foundation in the next section. Section 3 presents a dynamic general equilibrium model with indivisible capital and predetermined prices. I present a model with a continuum of firms and a model with a finite number of firms. The equilibrium dynamics for the former model coincides with the expectation system for the latter model with approximation. By numerically simulating the finite model, I show that equilibrium paths mimic the business cycles in the magnitude of standard deviations and correlations. Section 4 concludes. Most proofs are shown in the Appendix, and some detailed derivations are shown in a separate Technical Appendix accompanying this paper.

2 Analytical Results

In this section, I show the analytical results for aggregate fluctuations under a threshold behavioral rule. While the main focus is investment, the results shown here may apply for other fluctuation phenomena when a micro-behavior is characterized by a one-sided \((S,s)\) rule and complementarity. The behavioral rule and environments assumed in this section will be derived in a dynamic equilibrium model in the next section.

The synopsis of this analysis is as follows. An equilibrium capital profile
of $N$ firms can be reached by a fictitious tatonnement based on best-response dynamics, given a capital profile in the previous period that is drawn randomly from a stationary distribution of capital profile. Unconditional on the realization of the initial capital, the best-response dynamics can be embedded in the contact process described in Section 1. The number of firms affected in the contact process follows a branching process with the mean number of children born by a parent being equal to $\phi$. In a dynamic general equilibrium model, I show that $\phi = 1$ holds when goods prices are predetermined. When $\phi = 1$, it is known that the size of entire population that originated from one ancestor follows a power-law tailed distribution with exponent 0.5. This implies that the variance of the size of the entire population that originated from an ancestor scales as $N^{1.5}$. The number of ancestors corresponds in this model to the number of investing firms in the first step of tatonnement minus the expected number. This initial deviation obeys the central limit theorem, and thus the mean absolute initial deviation scales as $\sqrt{N}$. Thus, the variance of the net number of investing firms divided by $N$ scales as $N^{1.5}N^{0.5}/N^2$, which is non-vanishing in a large $N$. This property is robust even when investment size is heterogeneous across firms. When a firm’s investment size is larger, it has a proportionally greater impact on the aggregate capital. At the same time, the likelihood of this firm to be induced to invest by a shift in the aggregate capital is proportionally smaller. Thus, the mean impact of a firm’s possible investment on the aggregate capital remains the same across firms.

2.1 Setup for the analysis

2.1.1 Binary investment rule with complementarity

I consider the case where the firm’s capital choice is restricted to be binary: $k_{i,t+1} \in \{(1-\delta)k_{i,t}, \lambda_i(1-\delta)k_{i,t}\}$, where $\lambda_i(1-\delta) > 1$. Capital $k_{i,t+1}$ has to be either at the depreciated level $(1-\delta)k_{i,t}$ or at the depreciated level multiplied by the firm-specific indivisibility parameter $\lambda_i$. This binary constraint is equivalent to assuming that the firm can choose gross investment rate $x_{i,t}/k_{i,t}$ only either at 0 or $(\lambda_i - 1)(1-\delta)$, namely, inaction or lumpy investment, respectively. This constraint reflects the indivisibility of physical investment such as equipment and structure. It can also be interpreted as a shortcut for modeling the lumpy behavior, which typically occurs as an optimal investment policy under fixed adjustment costs. This paper is concerned
with the aggregate consequence of the non-linear behavior of firms induced by this indivisibility.

The firms’ investment choices exhibit complementarity with each other due to aggregate demand externality. Because of the binary restriction, a firm’s optimal behavior takes the form of a threshold rule. I use a simple behavioral rule for the threshold in this section as follows. Firm $i$ chooses a positive gross investment rate if and only if $(1 - \delta)k_{i,t}$ is below a certain threshold $k^*_i,t+1$, which is optimally determined as

$$k^*_i,t+1 = b(\lambda_i, a_{i,t+1})K_t^{\phi},$$

where $K_{t+1} \equiv \left(\sum_{i=1}^N a_{i,t+1}^{\rho/\alpha} k_{i,t+1}^{\phi}/N\right)^{1/\rho}$ and $a_{i,t+1}$ denotes firm-specific productivity. Parameter $\phi \in (0, 1]$ determines the strength of positive feedback from aggregate capital to individual investment decisions, and thus represents the degree of strategic complementarity between investments. This investment rule is derived as an optimal policy in a business cycle model in Section 3.

At the heart of the aggregate fluctuations arising from idiosyncratic shocks in this model lay the non-linearity and complementarity of firm-level investment decisions. The capital decision $k_{i,t+1}$ is non-linear because of indivisibility and threshold policy. Average capital level $K_{t+1}$ affects threshold $k^*_i,t+1$ continuously, but it may or may not induce an adjustment in capital $k_{i,t+1}$. Individual capital is insensitive to a small perturbation in average capital, while it synchronizes with average capital if the perturbation is large.

### 2.1.2 Random gap distribution

Equilibrium capital profile $(k_{i,t})_i$ must satisfy $k_{i,t} \geq k^*_i,t$. The gap between capital and threshold, normalized by indivisibility, is denoted by $s_{i,t} = (\log k_{i,t} - \log k^*_i,t)/\log \lambda_i$. In this section, I analytically derive the distribution of the fluctuations of aggregate capital $K_{t+1}$ when the initial capital profile $(k_{i,t})_i$ varies stochastically. Specifically, I draw $s_{i,t}$, which can be interpreted as individual capital relative to aggregate capital, from a distribution uniform over $[0, 1)$. In the business cycle model in Section 3, I will show that $s_{i,t}$ converges to the uniform distribution as $t \to \infty$, independent across $i$. This implies that the probability of drawing a particular profile $(s_{i,t})_i$ from an $N$-dimensional jointly uniform distribution corresponds to the likelihood of the profile being realized in the course of gap profile evolution in the long run.
Figure 1: Aggregate reaction function $\Gamma$. $K^1$ is selected by Equilibrium Selection 1 since $|\log K^1 - \log K_0^e| < |\log K^2 - \log K_0^e|$. $K^2$ is selected by Equilibrium Selection 2 as $\text{sign}(\log K^2 - \log K_0^e) = \text{sign}(\log \Gamma(K_0^e) - \log K_0^e)$.

2.1.3 Equilibrium selection

For each realization of gap and productivity profile $(s_{i,t}, a_{i,t+1})_i$, and given aggregate capital $K_{t+1}$, capital profile in the next period $(k_{i,t+1})_i$ is determined by (1). Then, an aggregate reaction function is defined as $K' = \Gamma(K; (k_{i,t}, \lambda_i, a_{i,t+1})_i)$ by aggregating the firms’ capital decision given $K$. $K$ enters $\Gamma$ via threshold rule (1). As depicted in Figure 1, $\Gamma$ is a step function, non-decreasing in $K$. Equilibrium aggregate capital is a fixed point of this reaction function.

The distribution function of the growth of $K_t$ is determined by the joint distribution function of $(s_{i,t}, a_{i,t+1})$, if this mapping is one-to-one. However, multiple solutions for (1) may exist as depicted in Figure 1, due to the indivisibility of capital. Thus, to obtain the distribution for fluctuations of $K_{t+1}$, an equilibrium selection mechanism is required.

I select—as an equilibrium—the fixed point of $\Gamma$ that is closest to the expected aggregate capital $K^e_{t+1} = E(K_{t+1} \mid K_t)$, which is determined in a business cycle model in Section 3. By this selection, I construct the least-possible volatile fluctuations of aggregate capital in equilibrium. In other
words, fluctuations due to multiple equilibria are excluded from the selected equilibrium. This is a strategic assumption I make in this paper in order to demonstrate that idiosyncratic shocks with non-linear behaviors alone can generate non-vanishing aggregate fluctuations even when I exclude the possibilities of big jumps that arise from a purely informational coordination among firms.

**Equilibrium Selection 1** For each initial capital vector \((k_{i,t})\), pick the equilibrium aggregate capital \(K_{t+1}^e\) that attains the minimum of \(\left| \log K_{t+1} - \log K_{t+1}^e \right|\) among all \(K_{t+1}\) that solve (1).

Expected aggregate capital \(K_{t+1}^e\) is a function of aggregate state \(K_t\). In the full business cycles model in the next section, model agents form expectations by the approximated gap distribution that is uniform over unit interval. Realization of a finite-length vector \((s_{i,t})_i\) necessarily deviates from the uniform distribution. This deviation from the uniform distribution corresponds to the deviation of the aggregate reaction function \(\Gamma\) from the 45 degree line in Figure 1. This deviation is quite small when \(N\) is large. Nonetheless, I show below that the difference between equilibrium aggregate capital and the expected one persists even when \(N\) tends to infinity.

To facilitate the analysis of this equilibrium, I define another selection mechanism as an auxiliary.

**Equilibrium Selection 2** For each initial capital vector \((k_{i,t})\), pick the equilibrium aggregate capital \(K_{t+1}^e\) that attains the minimum of \(\left| \log K_{t+1} - \log K_{t+1}^e \right|\) among all \(K_{t+1}\) that solve (1) and satisfy \(\text{sign}(\log K_{t+1} - \log K_{t+1}^e) = \text{sign}(\log \Gamma(K_{t+1}^e) - \log K_{t+1}^e)\).

This mechanism selects the equilibrium aggregate capital that is closest to the initial aggregate capital in the direction toward which the firms are induced to adjust by the expected aggregate capital. In Figure 1, this mechanism selects \(K^2\). Vives (1990) showed that the equilibrium selected by this mechanism can be reached as a convergent point of the best-response dynamics \(K_{t+1} = \Gamma(K_t)\) starting at \(K_0^e\). Cooper (1994) supported the use of this selection mechanism in macroeconomics on the grounds that the best-response dynamics is a realistic tatonnement process in a situation where many agents interact with each other. The only information needed for an agent to make decisions in the tatonnement is the aggregate capital level.
I first analyze the fluctuation of the equilibrium selected by the second mechanism, and then proceed to analyze the one selected by the first mechanism. I mostly concentrate on a homogeneous setup in which indivisibility and productivity are common across firms: $\lambda_i = \lambda$ and $a_{i,t} = 1$. Generalization to the case of heterogeneous indivisibility is discussed in Section 2.2.4. In this homogeneous setup, the only source of deviation from the expected aggregate capital is gap $s_{i,t}$.

To summarize, this section characterizes the fluctuations of $K_{t+1}$ provided that the investment follows threshold rule (1), the gap $s_{i,t}$ follows uniform distribution, and the expectation $K^e_{t+1}$ is well defined. Those three premises will be shown as equilibrium properties in Section 3.

2.2 Results

2.2.1 Distribution of aggregate capital growth rate

The equilibrium aggregate capital growth rate, $\log K_{t+1} - \log K_t$, consists of an anticipated part $\log K^e_{t+1} - \log K_t$ and an unanticipated part $g_{t+1} \equiv \log K_{t+1} - \log K^e_{t+1}$. The former part is deterministic, since the expectation system determines $K^e_{t+1}$ given $K_t$. I focus on the distribution of the unanticipated growth $g_{t+1}$. I introduce a notation $q_t \equiv \log \lambda / (\phi(\log K^e_{t+1} - \log K_t) - \log(1 - \delta))$. This is an inverse of the anticipated shift in $k^*_t$. At the steady state, $q$ represents the natural frequency of a firm’s capital adjustment. Henceforth, I drop the time subscript $t$ from all variables, and focus on $g$ given expected capital $K^e_0$.

Unanticipated growth $g$ is divided into two parts: adjustments in the initial round of fictitious tatonnement and subsequent adjustments. The first part, measured as the number of firms, is denoted as $m_1 \equiv N(\log \Gamma(K^e_0) - \log K^e_0)/\log \lambda$ (see Figure 1). If $m_1 = 0$, then $K^2 = K^e_0$ constitutes the equilibrium aggregate capital selected by mechanism 2. Otherwise, $K^2 \neq K^e_0$.

I state the main technical result here.

**Proposition 1** Under equilibrium selection 2, $Ng$ converges in distribution to $(m_1 + M) \log \lambda$, where $M$ conditional on $m_1 > 0$ follows:

$$\Pr(M = m \mid m_1) = m_1 e^{-\phi(m+m_1)} \phi^m(m + m_1)^{m-1}/m!$$

for $m = 0, 1, \ldots$. Conditional on $m_1 < 0$, $-M$ follows the same distribution with $|m_1|$ instead of $m_1$. The tail of distribution (2) is approximated by

$$\Pr(M = m \mid m_1) \sim (m_1 e^{(1-\phi)m_1}/\sqrt{2\pi}) e^{-(\phi-1-\log \phi)m_1} m^{-1.5}. \quad (3)$$
$m_1/\sqrt{N}$ asymptotically follows a normal distribution with mean zero and variance $\sigma_1^2 = (1 - \lambda^{-2\rho/q})/(2\rho \log \lambda) - ((1 - \lambda^{-\rho/q})/(\rho \log \lambda))^2$.

The proof is deferred to Appendix C. Here, I outline the proof. I use a fictitious tatonnement as a workhorse for characterizing the aggregate fluctuations. The fictitious tatonnement is defined by the best-response dynamics of capital profile:

$$k_{i,1} = \begin{cases} 
\lambda(1 - \delta)k_{i,0} & \text{if } (1 - \delta)k_{i,0} < k_{i,0}^*, \\
(1 - \delta)k_{i,0} & \text{otherwise}
\end{cases} \quad (4)$$

$$k_{i,u+1} = \begin{cases} 
\lambda k_{i,u} & \text{if } k_{i,u} < k_{i,u}^* \\
k_{i,u}/\lambda & \text{if } k_{i,u} \geq \lambda k_{i,u}^* \\
k_{i,u} & \text{otherwise}
\end{cases} \quad (5)$$

where $K_u = (\sum_i k_{i,u}^0/N)^{1/\rho}$ and $k_{i,u}^* = bK_u^\rho$. Subscript $u$ represents a step in the fictitious tatonnement. Note that the best-response dynamics is consistent with the aggregate response function $K_{u+1} = \Gamma(K_u)$.

The expected number of firms that adjust capital in the first step is $N/q$. Their investments may not exactly balance with aggregate capital depreciation: $\Gamma(K_0^e)$ may not coincide with $K_0^e$. If not, the optimal threshold is updated under new aggregate capital and the adjustments in the second step take place. This procedure is iterated until there are no more firms that newly adjust.

Subsequent adjustments after the first step are measured in the number of firms that adjust capital upward in step $u$, denoted by $m_u$ for $u = 2, 3, \ldots, T$. If firms adjust downward (i.e., some firms that decide to invest in the first step, retract), $m_u$ is set as negative. Series $m_u$ are either positive or negative for all $u$ depending on $m_1 > 0$ or $m_1 < 0$. $M = \sum_{u=2}^T m_u$ denotes the total number of firms that adjust capital subsequently after the first step of tatonnement. $T$ is the stopping time of the tatonnement: $T = \min_{u;m_u=0} u$. The equilibrium capital vector is determined by the convergent point of the dynamics, $k_{i,T}$. $m_1 + M$ indicates the total deviation of the investment from the stationary level in terms of the number of firms.

In the first step toward Proposition 1, I show that the capital growth rate is asymptotically proportional to the number of firms that adjust.

**Lemma 1** $N(\log K_{u+1}^e - \log K_u^e)$ converges to $m_{u+1} \log \lambda$ as $N \to \infty$ almost surely for $u = 1, 2, \ldots, T - 1$. 

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The proof is in Appendix A. Lemma 1 implies that \( N \log K^2 - \log K_0 \to (m_1 + M) \log \lambda \). Thus, the computation of \( g \) reduces to counting the total number of investing firms net of the expected number of investing firms. I then show that the number of adjusting firms in the tatonnement asymptotically follows a Poisson branching process.

**Lemma 2** \( m_u \) for \( u = 2, 3, \ldots, T \) asymptotically follows a branching process, in which each firm in \( m_u \) bears firms in step \( u + 1 \) whose number follows a Poisson distribution with mean \( \phi \).

The proof is in Appendix B. A branching process is an integer stochastic process of a population in which each parent in a generation bears a random number of children in the next generation. In a Poisson branching process, the number of children borne by a parent is a Poisson random variable. It is known that a branching process converges to 0 in a finite time period with probability 1 if the mean number of children borne by a parent is less than or equal to 1 (Feller, 1957, p.276). This fact confirms that the best-response dynamics stops in a finite time \( T \) with probability 1 when \( \phi \leq 1 \). Thus, the best-response dynamics is a valid algorithm of equilibrium selection even when \( N \to \infty \). Moreover, the cumulative population size of the Poisson branching process is known to follow a Borel-Tanner distribution (Kingman, 1993, p.68). By combining the Borel-Tanner distribution with the Poisson distribution for \( m_2 \), I obtain the desired distribution (2).

By Stirling’s formula, the tail of (2) is approximated by (3). (3) shows that \( g \) conditional on \( m_1 \) asymptotically follows a gamma-type distribution that combines a power function \( m^{-1.5} \) and an exponential function \( e^{-(\phi - 1 - \log \phi)m} \). Note that \( \phi - 1 - \log \phi > 0 \) for \( \phi < 1 \). Since an exponential function declines faster than a power function, the tail distribution is dominated by the exponential when \( \phi < 1 \). Thus, the degree of strategic complementarity \( \phi \) determines the speed of the exponential truncation of the distribution.

\( \phi = 1 \) holds in the business cycles model in Section 3, and in this case, the distribution (3) becomes a power-law distribution with exponent 0.5. Whether the tail obeys an exponential decay or a power decay has an important implication for the moments of the distribution. If the tail decays exponentially, any \( k \)-th moment exists, because \( \int_0^\infty x^k e^{-x} dx \) is a gamma function and thus finite. To the contrary, if the tail decays in power with exponent \( \alpha \), only the moments lower than \( \alpha \) exist, since \( \int_0^\infty x^k e^{-\alpha x} dx \) is finite only for \( k < \alpha \). When the exponent of the power law is 0.5, even the mean diverges.
The power-law tail of the propagation effect, resulted from the criticality condition \( \phi = 1 \), generates the macro-level fluctuation. When this condition is not met, the aggregate fluctuations eventually die down as the number of firms increases to infinity. This is because \( \phi \), the mean number of children per parent, determines the trend population growth in the branching process. The mean population of the \( n \)-th generation is \( \phi^n \) given a single initial parent. The population diverges to infinity when the process is supercritical, \( \phi > 1 \), whereas the population decreases to zero if subcritical, \( \phi < 1 \). At the critical point \( \phi = 1 \), the population of a generation decreases to zero with probability 1 and yet the mean cumulative population diverges to infinity.

### 2.2.2 Aggregate fluctuations with an arbitrarily large \( N \)

The distribution of \( M \) conditional on \( m_1 \) has a pure power-law tail when \( \phi = 1 \). With exponent 0.5, the power-law distribution does not have either mean or variance. The conditioning variable \( m_1 \), which represents the initial deviation from expected capital in the tatonnement, obeys the law of large numbers and its variance decreases linearly in \( N \). These two effects cancel out in the unconditional variance of \( (m_1 + M)/N \), as I state in the following proposition.

**Proposition 2** The variance of \( g \) converges to a non-zero constant as \( N \to \infty \) when \( \phi = 1 \). The limit standard deviation is \( (\log \lambda) \sqrt{(2/\pi)(\sigma_1 + 1/3)} \sigma_1 \).

The proof is deferred to Appendix D. The main idea is as follows. Proposition 1 showed that \( m_1/\sqrt{N} \) asymptotically follows a normal distribution with finite variance. This implies that the mean of absolute value \( |m_1| \) scales as \( \sqrt{N} \). Proposition 1 also showed that \( Ng/\log \lambda - m_1 \) conditional on \( m_1 = 1 \) follows a power-law distribution with exponent 0.5 if \( \phi = 1 \). Then, the variance of \( Ng \) conditional on \( m_1 = 1 \) diverges as \( N^{1.5} \), because \( \int_{N} x^2 x^{-1.5} dx \sim N^{1.5} \). Combining these two results, I obtain that \( Ng \) unconditional on \( m_1 \) has variance scaling as \( N^2 \), since \( Ng \) can be divided into \( \sqrt{N} \) sub-population sets, each of which has variance that scales as \( N^{1.5} \). Hence, the variance of \( g \) scales as \( N^0 \).

The argument above shows that the power-law distribution is essential in obtaining scale-invariant fluctuations for \( g \). The key environment for the power law, \( \phi = 1 \), can be interpreted as perfect complementarity of indivisible investments. By perfect complementarity I mean that a proportional increase
in capital of all the other firms induces the same proportional increase in the capital of a firm, if the increment is much larger than the indivisibility. Due to the indivisibility of capital, however, a shock smaller than indivisibility does not cause a symmetric movement across firms. Thus, the firm’s investment behavior at criticality may be summarized as local inertia combined with global perfect complementarity.

It might appear counterintuitive that the aggregate variance does not converge to zero when there are only idiosyncratic discrepancies in the initial capital gap. Note that in a smoothly-adjusting, competitive economy, the aggregate capital level is indeterminate in the production sector if the firms’ investment decisions are perfect complements due to the constant returns to scale technology. In the present model, the equilibrium is locally unique because of the indivisibility of capital. Nonetheless, the globally indeterminate environment makes it possible for the aggregate fluctuations to reappear in the form of the power-law distribution.

The limit standard deviation of \( g \) in Proposition 2 is determined by indivisibility parameter \( \lambda \) and periodicity \( q \) of capital oscillation at the firm level. I note that the indivisibility parameter \( \log \lambda \) has an almost proportional effect on the aggregate standard deviation. The standard deviation is a product of \( \log \lambda \) and a function of \( \sigma_1 \), which depends on \( \lambda^\rho \). It turns out that \( \sigma_1 \) shows little dependence on \( \rho \), which is determined by the markup rate in the general equilibrium model. In fact, the standard deviation does not significantly change even when the markup rate goes to infinity, at which \( \sigma_1^2 \) is simplified to \((1 - 1/q)/q \). The proportional impact of \( \log \lambda \) on the aggregate standard deviation can be also seen in numerical simulation, as in Table 2. This implies that the indivisibility of capital provides a foundation for the sizable idiosyncratic volatility of the firm-level decisions, which in turn has one-to-one impact on the aggregate volatility.

### 2.2.3 Case of Equilibrium Selection 1

Next, I investigate the magnitude of the fluctuation in aggregate capital selected by equilibrium selection 1. This selection picks the least-possible volatile equilibrium path, and will be used in the business cycle simulations in the next section. Let \( g^1 \) denote the capital growth rate under this selection. I obtain the following result.

**Proposition 3** Under equilibrium selection 1, the convergence of the variance of \( g^1 \) to zero as \( N \to \infty \) is not faster than \( 1/\sqrt{N} \) if \( \phi = 1 \).
Proof is in Appendix E. Proposition 3 shows that if I choose the least-volatile equilibrium, the variance of the capital growth rate decreases to zero as \( N \) increases, but at a rate much slower than what the central limit theorem predicts. In the Long-Plosser model with continuous capital adjustments, idiosyncratic technological shocks cancel out in aggregation and the aggregate variance declines as fast as \( 1/N \) (Dupor (1999)). In contrast, in this model, the variance of \( g^i \) declines no faster than \( 1/pN \).

This again opens up a theoretical possibility that indivisible investment at the micro level contributes to sizable macro-level fluctuations when the number of firms is large but finite.

Table 2 shows the standard deviations in the numerical simulations of equilibrium paths when the number of firms is 100000, about one-third of the benchmark case 350000. Standard deviations exhibit no discernible increase from the benchmark. Thus, in the calibrated range of parameter values, the number of firms does not have a significant impact on the magnitude of aggregate fluctuations.

2.2.4 Heterogeneous firms

In this section, I extend the fluctuation results to the case where both the indivisibility and depreciation rates are heterogeneous across firms. Suppose that a type of firm with lumpiness \( \lambda_i \) and depreciation \( \delta_i \) is drawn from a joint density function. The lower bound of the inaction band becomes heterogeneous as in (44): \( k_i^* = b_i K^\phi \). I maintain that productivity \( a_{i,t} \) is homogeneous across \( i \). Define \( \log \lambda_i \equiv b_i^* (\lambda_i^\phi - 1)/(\rho E[b_i^* \lambda_i^{s_t,1/\phi}]) \). Then, I obtain the following proposition.

**Proposition 4** \( M \) conditional on \( m_1 = 1 \) follows the same tail distribution as (3):

\[
\Pr(|M| = m \mid m_1 = 1) = C_0(e^{\hat{\phi} - 1}/\hat{\phi})^{-m}m^{-1.5}
\]

for a large integer \( m \), where \( \hat{\phi} \equiv \phi E[\log \lambda_i/\log \lambda_i] \) and \( C_0 \) are constant. The asymptotic variance of the fraction of firms that adjust, \( (m_1 + M)/N \), is strictly positive when \( N \to \infty \) if \( \hat{\phi} = 1 \).

Proof is in Appendix F. Proposition 4 shows that the power-law tail distribution with the same exponent is obtained even in the general setup where the indivisibility and depreciation rates are heterogeneous across firms. This is an important generalization for the business cycles model, as empirical studies attest large variations in the lumpiness in investment-capital ratio across firms (Doms and Dunne (1998); Cooper, Haltiwanger, and Power
(1999)). It is also a necessary extension for this paper, because the uniform distribution of $s_{i,t}$ is proved when $\lambda_i$ has a non-trivial density in the business cycle model in Section 3.\textsuperscript{1}

2.3 Relation to previous research

The power-law tail with exponent 0.5 characterizes the aggregate fluctuations even in a heterogeneous extension of the model (Section 2.2.4). The robustness of the exponent 0.5 results from the fact that any branching process with martingale property (i.e., $\phi = 1$) brings out the power-law tail with exponent 0.5 for the cumulative population size (Harris, 1989, p.32).\textsuperscript{2} The robustness reflects the fact that in various models of connected non-linear dynamics, the perfect complementarity $\phi = 1$ appears as a condition for idiosyncratic shocks to have aggregate consequences through power-law distributions. For example, in a celebrated theorem by Erdős and Rényi, the condition $\phi = 1$ corresponds to the critical point for the emergence of a “giant cluster” in a random graph (Bollobás, 1998, p.240).

In the literature on economic fluctuations, Jovanovic (1987) demonstrated in several simple models that aggregate fluctuations could be generated by interactions of idiosyncratic shocks. Notably, he pointed out that a key condition for the aggregate risks to emerge from interacting idiosyncratic shocks is that “the effect that a unit increase in the average decision of others has on [an individual decision]” is 1. This corresponds to the perfect complementarity condition $\phi = 1$. He shows some examples in which a “multiplier” effect of an individual’s action has the order of magnitude $\sqrt{N}$. The multiplier blows up the idiosyncratic shocks that shrink in aggregation as $1/\sqrt{N}$, and thus generates non-vanishing aggregate fluctuations. In the case of adjustments in extensive margin as featured in this model, the propagation effect (how many firms are affected) becomes stochastic rather than a constant multiplier. This paper develops Jovanovic’s insight and fully characterizes the fluctuations in extensive margin. The analysis above showed that the variation of the propagation effect, rather than the mean, has the order of

\textsuperscript{1}When random productivity is incorporated, it is still possible to characterize the fluctuation in the form of a moment generating function (Nirei (2006)), but it becomes difficult to derive the distribution function analytically.

\textsuperscript{2}The distribution of population size in branching processes is closely related to the distribution of the first return time of a random walk, which has the same power-law exponent 0.5.
magnitude $\sqrt{N}$.

In a general model of industries with binary technological choice and complementarities, Durlauf (1993) showed that the degree of complementarities determines whether an economy has a unique equilibrium or multiple equilibria. The present model is narrower than his in a sense that the firm’s behavioral rule is parametrically specified and the firms interact only through aggregate capital. The analysis here, however, differs in its aim. By specifying an equilibrium selection mechanism, this model excludes the fluctuations from multiple equilibria and concentrates on the least-volatile fluctuations. While Durlauf’s paper explains long-run phenomena such as industrialization, this paper concerns with short-run fluctuations such as business cycles. With a narrowly setup model, this paper delivers a sharp characterization of the distribution of aggregate fluctuations.

The possibility of a power-law distribution of sectoral propagation was first pointed out by Bak, Chen, Scheinkman, and Woodford (1993). In a model of simple supply chain with lattice network, they obtained a power-law distribution of aggregate fluctuations. My previous work (Nirei (2006)) implemented their fluctuation mechanism in an equilibrium model of globally connected network where an agent’s action affects all the other agents. A fluctuation distribution similar to (2) was obtained in the paper. The present paper extends the previous one by proving the non-zero asymptotic variance of the aggregate growth rate for the case of $\phi = 1$. The mechanism for the break of the law of large numbers is analogous to Jovanovic’s: the variation of the extensive margin (the number of firms affected) implied by distribution (2) turns out to be $\sqrt{N}$, which cancels out with the shrinking magnitude of initial disturbance due to depreciation by the law of large numbers as $1/\sqrt{N}$. In addition, this paper is placed under a standard real business cycles framework (Section 3), which underpins three key assumptions in this section: the perfect complementarity $\phi = 1$, the uniform distribution of gap $s_{i,t}$, and the well-defined expectation formation $K_{t+1}^e$.

This model may be classified as a self-organized criticality model as ad-

\footnote{A corollary difference between these two papers occurs in the exponent of the power-law distribution, which arises from the different network topology. Their model features a two-dimensional lattice network in which two avalanches starting from neighboring sites can overlap. This leads to a longer chain reaction and a lower power-law exponent (1/3). In contrast, a market equilibrium model features an essentially dimensionless network of firms. Thus, the market model corresponds to an infinite-dimension case of lattice models, which yields the cluster-volume exponent 0.5 at criticality (Grimmett, 1999, p.256).}
vocated by Bak, Tang, and Wiesenfeld (1987). In that interpretation, the
criticality in this model is the uniform distribution of $s_{i,t}$. When the density
of $s_{i,t}$ at the threshold is greater than 1, large propagation of investments
ensues. When the density at threshold is smaller than 1, little propagation
occurs. In either case, diffusion effects caused by heterogeneous indivisibility
$\lambda_i$ and productivity $a_{i,t}$ bring the density at the threshold to 1, at which the
size of propagation follows a power-law distribution. However, the key condi-
tion $\phi = 1$ is not self-organized but set exogenously. This paper claims that
the aggregate fluctuations arise from idiosyncratic shocks in an environment
where individual decisions are perfectly complementary, that is, when $\phi = 1$
is realized. While this is a restrictive condition, there are several important
examples that satisfy this condition in an economy, as shown by Jovanovic
(1987). In the next section, I show another example in investment decisions,
where the perfect complementarity condition is satisfied if goods prices are
predetermined. I interpret this to imply that the fluctuation mechanism con-
sidered in this particular model works only in the short run but not in the
long run.

3 A Business Cycle Model

In this section, I construct a dynamic general equilibrium model with indivis-
ible investments and predetermined prices of goods. I first present a model
with a continuum of firms, and then present a model with a finite number
of firms. The equilibrium system of the continuum economy turns out to
coincide with the expectation system in the finite economy.

3.1 Continuum economy

3.1.1 Households

There is a representative household with King-Plosser-Rebelo preference.
The representative household maximizes utility

$$E_t \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} C_{\tau}^{1-\sigma} (1 - \psi L_{\tau}^{\xi})^{1-\sigma} / (1 - \sigma) \right]$$

by choosing consumption $C_{\tau}$ and labor supply $L_{\tau}$ subject to $C_{\tau} = w_{\tau} L_{\tau} + D_{\tau}, \forall \tau$. $D_t$ denotes aggregate dividends that the household receives from
firms. Each firm $i$ owns capital and delivers dividend $d_{i,t}$. Households as shareholders instruct each firm to maximize its expected discounted sum of dividends stream $\sum_{\tau=t}^{\infty} \Delta_{t,\tau} d_{i,\tau}$. The discount factor is $\Delta_{t,\tau} \equiv \prod_{s=t+1}^{\tau} R_s^{-1}$, where $\Delta_{t,t} = 1$ by convention, and $R_t$ is the inverse of a stochastic discount factor

$$R_t^{-1} \equiv \beta C_t^{-\sigma}(1 - \psi L_t^\zeta)^{1-\sigma}/(C_{t-1}^{-\sigma}(1 - \psi L_{t-1}^\zeta)^{1-\sigma}). \quad (7)$$

The representative household consumes a composite consumption good that is produced by a CES function $C_t = \left( \int z_{c,i,t} \right)^{\eta/(\eta-1)}$, where $\eta > 1$ is the elasticity of substitution. The first-order conditions with respect to $C_t$ and $L_t$ yield:

$$w_t = C_t^\psi \xi L_t^{\zeta-1}/(1 - \psi L_t^\zeta). \quad (8)$$

### 3.1.2 Firms

In this section, I suppose that there are a continuum of firms indexed by $i \in [0,1]$. Each firm has a Cobb-Douglas production function with constant returns to scale: $y_{i,t} = a_{i,t} k_{i,t}^{1-\alpha}$. Productivity $a_{i,t}$ is a random variable, independently and identically distributed across $i$ and $t$. I assume that all firms know the realization of the productivity profile $(a_{i,t})$ in period $t-1$. I also assume that the support of $\log a_{i,t}$ is bounded so that $\Pr(|\log a_{i,t} - \log a_{i,t-1}| < |\log (1 - \delta)\alpha (1 - \rho)| = 1$, where $\delta$ and $\rho$ are parameters as defined below.

Firm $i$ produces good $i$ monopolistically, and it faces demand function $y_{i,t} = p_{i,t}^\eta Y_t$, where $Y_t \equiv \left( \int y_{i,t}^{\eta/(\eta-1)} dx_{i,t} \right)^{\eta/(\eta-1)}$ denotes the aggregate output. Firm $i$ owns physical capital $k_{i,t}$, which accumulates as $k_{i,t+1} = (1 - \delta)k_{i,t} + x_{i,t}$. The investment good is produced by the CES function similar to the consumption good: $x_{i,t} = \left( \int z_{i,j,t}^{\eta/(\eta-1)} dx_{j,t} \right)^{\eta/(\eta-1)}$. The dividend paid by $i$ is $d_{i,t} = p_{i,t} y_{i,t} - w_t k_{i,t} - x_{i,t}$. The aggregate dividend is $D_t = \int d_{i,t} dx_{i,t}$. I assume that firm $i$ commits to the price of its product $p_{i,t}$ one period ahead. Namely, firm $i$ decides $p_{i,t}$ in period $t - 1$. The aggregate price $(\int p_{i,t}^{1-\eta} dx_{i,t})^{1/(1-\eta)}$ is normalized to 1.\footnote{This normalization is innocuous, because all the firms decide the goods prices simultaneously. Real wage $w_t$ is flexible.} The time protocol is set as follows. At the beginning of period $t$, productivities $a_{i,t+1}$ are revealed to all firms. Next, firm $i$ decides its price $p_{i,t+1}$ and capital $k_{i,t+1}$ of the next period, while
next-period aggregate capital $K_{t+1}$ and contemporaneous investment $X_t$ are determined simultaneously. Finally, contemporaneous $y_{i,t}$, $l_{i,t}$, $d_{i,t}$, and $C_t$ are determined given $X_t$.

Firm $i$’s problem in $t-1$ is to maximize $E_{t-1}\left[\sum_{\tau=t-1}^{\infty} \Delta_{\tau-1} \Delta_{\tau} d_{i,\tau}\right]$ by choosing $p_{i,t}$ and $k_{i,t}$ subject to the demand function, production function, and binary constraint for capital. The optimal price is solved by maximizing the dividend in $t$ as $p_{i,t} = (a_{i,t}^{1/\alpha} k_{i,t}/K_t)^{-\rho/\alpha/(\eta(1-c_1))}$, where $c_1 = (1 - 1/\eta)(1 - \alpha)$, $\rho = (1 - 1/\eta)/(1 - c_1)$, and $K_t = \left(\int a_i^{\rho/\alpha} k_{i,t}^{\rho} d_i\right)^{1/\rho}$. Substituting $p_{i,t}$ in the demand function and aggregating across $i$, I obtain

$$K_t = \left(E_{t-1}[w_t/c_1] Y_t^{1/(1-\alpha)} / R_t] / E_{t-1}[Y_t / R_t]\right)^{(1-\alpha)/\alpha}. \quad (9)$$

By using the optimal price, firm $i$’s problem in $t-1$ boils down to choosing $k_{i,t}$ from a binary set $\{(1-\delta)k_{i,t}, \lambda_i(1-\delta)k_{i,t}\}$, as in the previous section, to maximize $\pi(k_{i,t}) = (1-c_1)E_{t-1}[Y_t / R_t]a_k^{\rho/\alpha}(k_{i,t}/K_t)^{\rho} - (1 - (1-\delta)E_{t-1}[R_t^{-1}])k_{i,t}$. The optimal strategy for firm $i$ is to invest in $t-1$ when $(1-\delta)k_{i,t-1}$ is below a threshold level $k_{i,t}^*$, and not to invest otherwise. At the threshold, the firm must be indifferent between investing and not investing. This implies $\pi(\lambda_i k_{i,t}^*) = \pi(k_{i,t}^*)$, because the capital stream after $t+1$ when investing in $t-1$ coincides with that when the investment is deferred to $t$. Threshold $k_{i,t}^*$ is uniquely and explicitly obtained by solving this equation, since $\pi$ is strictly concave since $\rho < 1$, as shown in Appendix G.

### 3.1.3 Market-clearing conditions

Consumption $C_t$ and investment $x_{i,t}$ are composite goods, and the derived demand for good $j$ is denoted by $z_{c,j}$ for a household and $z_{i,j}$ for firm $i$. Thus, the goods markets clear by $y_{j,t} = z_{c,j,t} + \int z_{i,j,t} di$, $\forall j$. Aggregating these under cost-minimization conditions, I obtain

$$Y_t = C_t + X_t, \quad (10)$$

where $X_t \equiv \int x_{i,t} di$ is aggregate investment.

The labor market clearing condition is $L_t = \int l_{i,t} di$. By substituting the price-setting rule in labor demand $l_{i,t} = (p_{i,t}^\eta Y_t/\left(a_{i,t} k_{i,t}^{\alpha}\right))^{1/(1-\alpha)}$ and aggregating, I obtain an aggregate production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}. \quad (11)$$
3.1.4 Capital gap distribution

As in Section 2, I define capital gap as \( s_{i,t} \equiv (\log k_{i,t} - \log k^*_{i,t}) / \log \lambda_i \). This gap \( s_{i,t} \) always takes a value between 0 and 1 at equilibrium. The gap develops as

\[
s_{i,t+1} = \left( \frac{\log(1 - \delta) + \log k^*_{i,t} - \log k^*_{i,t+1}}{\log \lambda_i} + s_{i,t} + 1 \right) \mod 1, \tag{12}
\]

where “\( x \mod 1 \)” denotes the remainder after division of \( x \) by 1. Starting from an initial state \( s_{i,0} \), \( s_{i,t} \) is given as natural depreciation \( t \log(1 - \delta) \) divided by \( \log \lambda_i \), plus a random variable, and taken modulo 1. When \( 1/\log \lambda_i \) has a density, this remainder converges to a uniform distribution on a unit interval (Engel, 1992, 3.1.1).

**Proposition 5** As \( t \to \infty \), \( s_{i,t} \) converges in distribution to a uniform random variable in \([0, 1)\).

The proof is in Appendix G. Proposition 5 corresponds to a robust feature of one-sided \((S,s)\) economies as shown in Caplin and Spulber (1987) and Caballero and Engel (1991). Note that the cross-section distribution of \( s_{i,t} \) stays at the uniform distribution even if aggregate variables fluctuate, since a shift in \( K_t \) merely rotates the distribution of \( s_{i,t} \) on a circle of unit circumference.

3.1.5 Aggregate variables under a stationary gap distribution

Since there are a continuum of firms, idiosyncratic shocks \( a_{i,t} \) are aggregated out and there is no aggregate risk in this economy. Thus, (9) is reduced to

\[
K_t = (w_t/c_1)^{1/\alpha} Y_t. \tag{13}
\]

I consider the case of a stationary gap distribution. Substituting the uniform distribution of \( s_{i,t} \) in the marginal cost condition (42) and using (13), I obtain a familiar condition on marginal costs in the constant returns to scale economy:

\[
1 = a^{\rho - 1}(1 - c_1)(w_t/c_1)^{-1/\alpha} (R_t - 1 + \delta)^{-1}, \tag{14}
\]

\[
a \equiv \left( \int \left( \frac{\lambda_i^\rho - 1}{(\lambda_i - 1)^\rho} \right)^{1/\rho} \frac{a_{i,t}^{1/1-\rho}}{\rho \log \lambda_i} \, di \right)^{-1/\rho}. \tag{15}
\]
Under the stationary distribution, the threshold becomes a function of only
the idiosyncratic productivity and the aggregate capital:

\[ \frac{k_{i,t}^*}{K_t} = b_{i,t} = a \left( a_{i,t}^\rho \lambda_{i,t}^\rho / (\lambda_{i,t} - 1) \right)^{1/(1-\rho)}. \] (16)

The threshold capital \( k_{i,t}^* \) can be translated to threshold gap \( s_{i,t}^* \), where
firms with \( s_{i,t} \in [0, s_{i,t}^*] \) invest in \( t \). Since \( a_{i,t+1} \) is known to \( i \) in \( t \), \( s_{i,t+1} = 0 \)
holds at \( s_{i,t} = s_{i,t}^* \). Thus, the threshold is obtained from (12) as

\[ s_{i,t}^* = \frac{\log k_{i,t}^* - \log k_{i,t}}{\log \lambda_i} - \frac{\log(1 - \delta)}{\log \lambda_i}. \] (17)

Due to the assumption of bounded increment of \( \log a_{i,t} \), gap \( s_{i,t} \) always
decreases over time unless there is an upward jump by 1. Aggregate gross
investment under the stationary uniform distribution of \( s_{i,t} \) is then written
as follows:\(^5\)

\[ X_t = \int_0^{s_{i,t}^*} (\lambda_i - 1)(1 - \delta)\lambda_{i,t}^{s_{i,t}^*} k_{i,t}^* ds_{i,t} di = \rho a_{i+1}^{1-\rho}(K_{i+1} - (1 - \delta)K_i). \] (18)

### 3.1.6 Equilibrium

I consider an economy in which capital gap \( s_{i,0} \) has achieved a stationary uniform
distribution across \( i \). An equilibrium consists of pricing functions \( w(K) \)
and \( R(K) \), the law of motion for \( K \), and aggregate allocation \( (Y, X, C, L, D) \)
such that the allocation solves the household’s problem given prices, that
the law of motion and the allocation are consistent with the firms’ optimal
investment policy, and that the goods and labor markets clear. The equilib-
rium path satisfies the system of equations (7, 8, 10, 11, 13, 14, 18). Bar
denotes the steady-state values. By log-linearizing the system around the
steady state, it is shown that the equilibrium path is locally determinate
under a mild condition.\(^6\)

**Proposition 6** There exists a unique saddle point path for the log-linearized
system of (7, 8, 10, 11, 13, 14, 18), if \( X/Y \leq \alpha \) holds.

\(^5\)See the Technical Appendix for a detailed derivation.
\(^6\)The proof is standard and provided in the Technical Appendix.
3.2 Finite economy

In this section, I turn to an economy where there are many but finite \( N \) firms instead of a continuum of firms. The economy experiences some fluctuations due to finite idiosyncratic shocks. I will show that the fluctuation of aggregate investment \( X_t \) remains non-trivial even when \( N \) is large. As before, I employ a predetermined price assumption, under which firm \( i \) commits itself to meeting the demand in \( t \) at price \( p_{i,t} \) that is decided in period \( t-1 \). When \( X_t \) differs from the expected level due to finite shocks, firms adjust their labor demand, and the labor market clears by adjusting the wage. Thus, under predetermined prices, the investment fluctuation causes quantity adjustments in hours worked, production, and consumption.

Aggregate variables are now redefined by averages such as

\[
Y_t = \frac{\sum_{i=1}^N y_{i,t}}{N}, \quad \frac{\sum_{i=1}^N z_{i,t}}{N}, \quad \frac{\sum_{i=1}^N x_{i,t}}{N}, \quad \frac{\sum_{i=1}^N a_{i,t}^{\rho/\alpha} k_{i,t}}{N}, \quad \frac{\sum_{i=1}^N p_{i,t}}{N}, \quad \frac{\sum_{i=1}^N l_{i,t}}{N}.
\]

The labor market clearing condition is

\[
L_t = \sum_{i=1}^N l_{i,t}/N.
\]

Similar to the continuum case, equilibrium conditions are derived as (7, 8, 9, 10, 11, 44) and

\begin{align}
K_{t+1} &= \left( \sum_{i: (1-\delta)k_{i,t} < k_{i,t+1}} \left( \frac{(\lambda_i - 1)k_{i,t}}{N} \right)^{\rho} + \sum_{i: (1-\delta)k_{i,t} \geq k_{i,t+1}} \left( \frac{(1-\delta)k_{i,t}}{N} \right)^{\rho} \right)^{\frac{1}{\rho}}, \\
X_t &= \sum_{i: (1-\delta)k_{i,t} < k_{i,t+1}} (\lambda_i - 1)(1-\delta)k_{i,t}, \\
1 &= \left( \sum_{i=1}^N a_{i,t}^{\rho(1-\rho)} \left( \frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\frac{\rho}{1-\rho}} \frac{\lambda_i^{\rho s_{i,t}}}{N} \right)^{\frac{1}{1-\rho}} B_{t-1}.
\end{align}

The state space involves the distribution of gap \( s_{i,t} \), which is included in the information set for the conditional expectation in period \( t \) and affects the summations in (19–21). The equilibrium is similar to that of Krusell and Smith, Jr. (1998) and is difficult to solve exactly. Thus, I approximate the equilibrium system by using the stationary distributions of \( s_{i,t} \) and \( a_{i,t} \) with a continuum of firms. By this approximation, the summations across \( i \) in (19–21) are replaced with integrals over the uniform distribution of \( s_{i,t} \). I assume that agents use this approximated equilibrium system to form expectations of future variables, whereas the exact realizations of \( k_{i,t}, K_{t+1}, X_t \)}
are determined by (16, 19, 20) while keeping summations. Then, the system of equations for the agents’ forecast becomes (7, 8, 9, 10, 11) and

\[1 = \frac{\alpha^{\rho-1}(1 - c_1)E_{t-1}[Y_t/R_t]^{\frac{1}{\rho}}}{(1 - (1 - \delta)E_{t-1}[R_t^{-1}])E_{t-1}[(w_t/c_1)Y_t^{1-\alpha}/R_t]^{1-\alpha}}, \tag{22}\]

\[K_{t+1}^e = (1 - \delta)K_t + (\alpha^{\rho-1}/\rho)X_t^e, \tag{23}\]

\[X_t = X_t^e e^{\epsilon_t}, \tag{24}\]

\[K_{t+1} = (1 - \delta)K_t + (\alpha^{\rho-1}/\rho)X_t = K_{t+1}^e + (\alpha^{\rho-1}/\rho)X_t^e(e^{\epsilon_t} - 1). \tag{25}\]

The aggregate investment demand shock \(\epsilon_t\) enters (24), defined as the log-difference between realized and expected investments.

The expectation system (7, 8, 9, 10, 11, 22, 23, 24, 25) can be approximated in the first order. The certainty equivalence of the log-linearized expectation system coincides with the log-linearized equilibrium system of a continuum economy. Thus, the expectation system has a determinate solution. Combined with \(\epsilon_t\), the equilibrium path fluctuates around the determinate saddle point path.

**Proposition 7** There exists a unique saddle point path for the expectation system if \(\bar{X}/\bar{Y} \leq \alpha\) holds.

The proof is provided in the Technical Appendix.

### 3.3 Investment demand shock

In a finite economy, the investment demand shock \(\epsilon_t\) is defined as a log-difference between realized aggregate investment \(X_t\) and expected aggregate investment \(E_{t-1}X_t\). \(X_t\) is determined along with \(K_{t+1}\) and \(k_{i,t+1}^e\) by (16, 19, 20) given exact capital \(k_{i,t}\) and realized productivity \(a_{i,t+1}\). \(E_{t-1}X_t\) is determined by the expectation system (7, 8, 9, 10, 11, 22, 23, 24, 25) given \(K_t\). The deviation of actual aggregate investment from the expected value is caused by idiosyncratic productivity shocks \(a_{i,t+1}\) for a finite number of firms and the deviation of the gap distribution from the uniform distribution.

Due to the non-linear decision of \(k_{i,t+1}\) with strategic complementarity across \(i\), there can be multiple solutions for (16, 19) for each state \((k_{i,t}, a_{i,t+1})_i\). For those cases, I use Equilibrium Selection 1 that picks the solution that minimizes \(|\epsilon_t|\) among all the solutions. Namely, this selection rule picks the
equilibrium path that minimizes the deviation from the expected equilibrium path determined by the continuum counterpart. In numerical simulations, $\epsilon_t$ is computed as follows. First, $\epsilon_t$ is set to 0, and $K_{t+1}$ and $k_{t,t+1}^*$ are computed given $k_{t,t}$. If $X_t$ under the threshold $k_{t,t+1}^*$ coincides with $E_{t-1}X_t$, then $\epsilon_t$ is determined at 0. Otherwise, $\epsilon_t$ is adjusted slightly, and the above procedure is repeated until the selected outcome is obtained.

## 3.4 Calibration and numerical simulations

For a benchmark calibration, I set the unit of time as a quarter. The parameters for technology and preference are set as in Table 1. Details on the calibration are deferred to Appendix H. Table 2 reports the standard deviations and comovement structure of output, consumption, investment, hours worked, and capital. As can be seen, the model is able to generate aggregate investment fluctuations to the magnitude comparable to the business cycles.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
<th>$\beta$</th>
<th>$\eta$</th>
<th>$\psi$</th>
<th>$\zeta$</th>
<th>$E(\lambda_t)$</th>
<th>$\text{Std}(\log a_{i,t})$</th>
<th>$N$</th>
<th>$\bar{wL}/\bar{Y}$</th>
<th>$\tilde{C}/\bar{Y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.26</td>
<td>0.02</td>
<td>1.5</td>
<td>0.99</td>
<td>1</td>
<td>2</td>
<td>1.028</td>
<td>0.05%</td>
<td>350000</td>
<td>0.67</td>
<td>0.84</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Benchmark calibration and endogenous steady-state values.

<table>
<thead>
<tr>
<th>$\sigma = 3$</th>
<th>$E[\lambda_t] = 1.056$</th>
<th>$N = 100000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{Y}$</td>
<td>$\tilde{C}$</td>
<td>$\tilde{X}$</td>
</tr>
<tr>
<td>2.23</td>
<td>0.85</td>
<td>6.44</td>
</tr>
<tr>
<td>(0.06)</td>
<td>(0.02)</td>
<td>(0.17)</td>
</tr>
<tr>
<td>$\bar{L}$</td>
<td>$K$</td>
<td></td>
</tr>
<tr>
<td>3.02</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>(0.08)</td>
<td>(0.01)</td>
<td></td>
</tr>
<tr>
<td>$\bar{C}$</td>
<td>$\tilde{X}$</td>
<td>$\tilde{L}$</td>
</tr>
<tr>
<td>0.820</td>
<td>0.979</td>
<td>0.793</td>
</tr>
<tr>
<td>(0.004)</td>
<td>(0.001)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>$\tilde{K}$</td>
<td>$\bar{Y}$</td>
<td></td>
</tr>
<tr>
<td>-0.020</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.005)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Standard deviations and correlations of business cycles variables.

The fluctuations in aggregate variables are driven mostly by investment demand shocks $\epsilon_t$, while movements in capital play little role. The investment demand shock $\epsilon_t$ propagates to other variables in two paths: $K_{t+1}$ and $Y_t$.

On one hand, an investment demand shock generates an exogenous increase
in future capital $K_{t+1}$. This raises future labor productivity and real wage. The prospect of increased labor productivity induces households to consume more in both the next period and the current period. This effect can be seen in the saddle point path in which the marginal utility of consumption is negatively related to capital. On the other hand, an increase in investment demand raises aggregate demand for contemporaneous goods, provided that consumption demand is unaffected. Firms respond to the increased demand by increasing labor demand, which raises real wage. Households respond to the higher real wage by raising hours worked, which in turn raises the marginal utility of consumption when $\sigma > 1$. Thus, in order to keep the marginal utility lower so that it is on the saddle point path, consumption demand must increase. Hence, the investment demand shock raises consumption, and thus, output. In Table 2, I observe that the standard deviation of consumption relative to investment is larger when $\sigma$ is greater. This result is consistent with the propagation mechanism described above, because the hours-consumption complementarity, given a fixed marginal utility of consumption, becomes larger when $\sigma - 1$ is greater.

The dynamic model provides a key mechanism for investment-consumption comovement as described above. Moreover, it generates the self-organization of gap distribution toward the uniform distribution at which power-law propagation effects emerge. It also underpins the key environments for the non-vanishing aggregate fluctuations: $\phi = 1$ and a well-defined $K^c_{t+1}$. However, the dynamic model in the current form does not generate quantitatively strong autocorrelation. The impulse-response function shows that even though the dynamic pattern described above (the one that generates the contemporaneous comovement) is qualitatively present, the convergence to stationary levels is so fast that little deviation is seen in one quarter after an impulse on investment.\(^7\) I leave it for future research to incorporate a mechanism that generates persistence.

### 3.5 Discussion on model assumptions

Assumptions important for aggregate fluctuations are capital indivisibility and predetermined prices of goods. When indivisibility $\log \lambda$ is lowered, Proposition 2 states that the asymptotic aggregate standard deviation decreases almost proportionally. This is because with little indivisibility, capital

\(^7\)See the Technical Appendix for the plot.
can closely keep track of idiosyncratic productivity, which cancels out across firms in aggregation.

The predetermined price is important for the criticality condition $\phi = 1$. Under the predetermined prices, firms are committed to accommodating demand shocks by output only. Thus, an increase in aggregate investment causes firms to hire more, which raises contemporaneous consumption. In contrast, under flexible prices, firms are able to increase their prices and suppress their outputs upon a demand shock. Thus, as in Thomas (2002), an increase in aggregate investment raises factor prices and dampens the investment demand. The key difference is that the efficient hiring condition (9) holds only in expectation in the predetermined prices model.

In the model, I also assume that firms use the stationary gap distribution in order to form expectations. In deciding $k_{t+1}$, firms observe $K_{t+1}$ and $X_t$ but not $X_{t+1}$. To form expectations on $w_{t+1}$ and $R_{t+1}$, firms assume that future gap distributions are uniformly distributed. This assumption is necessary to render the model tractable. Simulated gap distributions show little deviation from the uniform distribution. In Appendix I, I further examine that using the exact gap distribution does not significantly improve firms’ prediction power over future prices.

4 Conclusion

This paper characterizes the aggregate fluctuations arising from the complementarity of indivisible investments at the firm level. Analytically, I propose to evaluate the fluctuation of aggregate investment along the evolution of heterogeneous capital as if it is a stochastic fluctuation whose randomness arises from the stochastic configuration of relative capital levels. For each configuration, the equilibrium aggregate investment is determined as a convergent point of a fictitious best-response dynamics of firms’ investment decisions. The best-response dynamics can be embedded in a branching process with a probability measure of the stochastic configuration of relative capital. This enables us to derive the distribution function of the aggregate fluctuation in a closed form.

The fluctuation in the number of investing firms is shown to follow a power-law distribution with an exponential truncation at the tail. The truncation speed is determined by the degree of strategic complementarity among firms. In the model of predetermined price setting, the distribution becomes
a pure power law, and the standard deviation of the growth rate is shown to be strictly positive even when there are an infinite number of firms. The limiting standard deviation is shown to be almost proportional to the indivisibility of firm-level investments.

I incorporate the above fluctuation mechanism in a dynamic general equilibrium model and numerically compute equilibrium paths without making the randomness assumption of the capital configuration. Under plausible parameter values, the equilibrium path is shown to exhibit aggregate fluctuations comparable to business cycles in magnitude and correlation structure. The simulation also confirms the validity of the analysis above that utilizes the assumptions of randomness and uniformity of the capital configuration.

A Proof of Lemma 1

Let $H_u$, $u = 2, 3, \ldots, T$ denote the set of firms that adjust capital in step $u$. Assume that the size of $H_u$ is finite with probability one when $N \to \infty$, which I verify later. I consider the case $m_1 > 0$ for the proofs of Lemmas 1 and 2 and Proposition 1 without loss of generality. Thus, $\log k_{i,u} = \log k_{i,u-1} + \log \lambda$ for $i \in H_u$.

The Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ around $(\log k_u)_{i \in H_{u+1}}$ is calculated as follows. The first derivative is $\partial N \log K_u / \partial \log k_{i,u} = (k_{i,u}/K_u)^\rho$. Thus, $\partial K_u / \partial k_{i,u}$ is of order $1/N$. The second and higher derivatives with respect to own $\log k_{i,u}$ are $\partial^n (k_{i,u}/K_u)^\rho / \partial (\log k_{i,u})^n = \rho^n (k_{i,u}/K_u)^\rho + O(\partial K_u / \partial k_{i,u})$ for $n = 1, 2, \ldots$. The second cross derivatives, $\partial^2 \log K_u / (\partial \log k_{i,u} \partial \log k_{j,u})$, are of order $\partial^2 K_u / \partial k_{i,u} \partial k_{j,u}$, and thus, $O(1/N)$. Similarly, the higher-order cross derivative terms with respect to the capital of $h$ distinct firms in $H_{u+1}$ are of order $1/N^{h-1}$. Since $H_{u+1}$ is finite, the $n$-th derivative of $N \log K_u$ has a finite number of cross derivative terms for any finite $n$. Hence, the Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ yields

$$\sum_{n=1}^{\infty} \sum_{i \in H_{u+1}} \left( \frac{k_{i,u}}{K_u} \right)^\rho \frac{\rho^{n-1}(\log \lambda)^n}{n!} + O(1/N) = \frac{\lambda^\rho - 1}{\rho} \sum_{i \in H_{u+1}} \left( \frac{k_{i,u}}{K_u} \right)^\rho + O(1/N),$$

where I used $\lambda^\rho = \lambda^0 + \sum_{n=1}^{\infty} (d^n \lambda^\rho / d \rho^n)|_{\rho=0} (\rho^n/n!)$. Utilizing $k_{i,u} = k_u^{\rho} \lambda^{s_{i,u}}$, I obtain that $\sum_{i \in H_{u+1}} (k_{i,u}/K_u)^\rho = (\sum_{i \in H_{u+1}} \lambda^{s_{i,u}})^\rho / (\sum_{i=1}^{N} \lambda^{s_{i,u}})^\rho / (\sum_{i=1}^{N} \lambda^{s_{i,u}})^\rho / (\sum_{i=1}^{N} \lambda^{s_{i,u}})^\rho$. The

---

8By taking $y_N$ as of order $x_N$, or interchangeably, $y_N = O(x_N)$, I mean that $y_N/x_N$ converges to a finite number as $N \to \infty$. 
denominator converges to $E[\lambda_{s_{i,u}}]$ as $N \to \infty$ almost surely by the law of large numbers, and I have $E[\lambda_{s_{i,u}}] = \int_0^1 \lambda_{s_{i,u}} ds_{i,u} = (\lambda^\rho - 1)/(\rho \log \lambda)$. The numerator, $\sum_{i \in H_{u+1}} \lambda_{s_{i,u}}$, converges to $m_{u+1}$ for every event when $H_{u+1}$ is finite, because $s_{i,u}$ is smaller than $\phi(\log K_u - \log K_{u-1})/\log \lambda$ for any $i \in H_{u+1}$, and thus, $\lambda_{s_{i,u}}$ converges to 1 as $N \to \infty$. Thus, I obtain the lemma.

## B Proof of Lemma 2

The conditional probability for firm $i$ to invest in $u = 2, 3, \ldots, T$ is

$$\Pr(i \in H_u \mid i \notin \cup_{v=2,3,\ldots,u-1} H_v) = \frac{\phi(\log K_u - \log K_{u-1})/\log \lambda}{1 - \phi(\log K_{u-1} - \log K_0)/\log \lambda}.$$  \hfill (26)

Thus, $m_u$ follows a binomial distribution with population $N - \sum_{v=2}^{u-1} m_v$ and probability (26). The mean of $m_u$ converges to $\phi m_{u-1}$ as $N \to \infty$, by using Lemma 1. Then, the binomial distribution of $m_u$ converges to a Poisson distribution with mean $\phi m_{u-1}$ for $u = 2, 3, \ldots, T$. Since a Poisson distribution is infinitely divisible, the Poisson variable with mean $\phi m_{u-1}$ is equivalent to a $m_{u-1}$-times convolution of a Poisson variable with mean $\phi$. Thus, the process $m_u$ for $u = 2, 3, \ldots, T$ is a branching process with a Poisson random variable with mean $\phi$, where $m_2$ follows a Poisson distribution with mean $\phi m_1$. Note that $m_1$ is not included in the branching process because it is not necessarily an integer.

## C Proof of Proposition 1

The initial state of the best-response dynamics (4, 5) is constructed as follows. For the case $\phi < 1$, a gap profile $(s_{i,0})_i$ is drawn from a jointly uniform distribution, and a capital profile is constructed by $k_{i,-1} = \lambda_{s_{i,0}} K_{-1}^{\phi}$ and $K_{-1} = (\sum_{i=1}^N k_{i,-1}^{\phi}/N)^{1/\rho}$. For $\phi = 1$, the last equation determines $b_{-1}$ and $B_{-1}$ given the realized gaps, and $K_{-1}$ is determined by $B_{-1}$ in the expectation system. Then, the initial state of the tatonnement is constructed by $k_{i,0} = (1 - \delta)k_{i,-1}$ and by setting $K_0$ at $K_{0}^{c}$ that is determined by the expectation system given $K_{-1}$. In the notations for tatonnement, $\log K_T - \log K_{-1}$ corresponds to the capital growth rate in the model $\log K_{t+1} - \log K_t$.

I first derive the asymptotic distribution of $M$ conditional on $m_1$. It is known that the accumulated sum $M = \sum_{u=2}^T m_u$ of the Poisson branching
process conditional on $m_2$ follows an infinitely divisible distribution called the Borel-Tanner distribution (Kingman, 1993, p.68):

$$\Pr(M = m \mid m_2) = (m_2/m)e^{-\phi m}(\phi m)^{m-m_2}/(m - m_2)!$$  \hfill (27)

for $m = m_2, m_2 + 1, \ldots$. By combining (27) with $m_2$ that follows the Poisson distribution with mean $\phi m_1$, and using the binomial theorem in the summation over $m_2$, I obtain (2).\footnote{See the Technical Appendix for a detailed derivation.}

Furthermore, the approximation in (3) is obtained by applying Stirling’s formula $m! \sim \sqrt{2\pi}e^{-m}m^{m+0.5}$ and the fact that $(1 + m_1/m)^{m-1} \to e^{m_1}$ as $m \to \infty$.

Next, I derive the asymptotic normal distribution of $m_1/\sqrt{N}$. I split $m_1/\sqrt{N}$ into three terms as $\log \Gamma(K_0) - \log \left(\sum_{i=1}^N((1-\delta)k_{i-1})^\rho/N\right)^{1/\rho} - \log K_0$, and by the central limit theorem, the law of large numbers, $\Pr(H_0) \rightarrow 0$. Hence, the first term represents the depreciation and is equal to $(\sqrt{N}/\log \lambda) \log(1-\delta)$. Thus, the sum of the second and third terms yields $\sqrt{N}/q$. The first term represents the first-step adjustments induced directly by depreciation. Define $H_1$ as the set of firms that adjust in the first step. Using $k_{i,0} = \lambda s_i \theta k_0^*$, I obtain $K_1 = (1-\delta)k_0^*((\lambda^\rho - 1)\sum_{i\in H_1} \lambda s_i \theta \rho /N + \sum_{i=1}^N \lambda s_i \theta \rho /N)^{1/\rho}$ and $(\sum_{i=1}^N((1-\delta)k_{i,0})^\rho/N)^{1/\rho} = (1-\delta)k_0^*(\sum_{i=1}^N \lambda s_i \theta \rho /N)^{1/\rho}$. Hence, the first term of $m_1/\sqrt{N}$ becomes

$$\frac{\sqrt{N}}{\rho \log \lambda} \log \left(\frac{(\lambda^\rho - 1)\sum_{i\in H_1} \lambda s_i \theta \rho /N}{\sum_{i=1}^N \lambda s_i \theta \rho /N} + 1\right).$$  \hfill (28)

By assumption, $s_{i,0}$ is distributed uniformly. Thus, the denominator $\sum_{i=1}^N \lambda s_i \theta \rho /N$ in (28) converges to $\int_0^1 \lambda s_i \theta \rho \, ds_{i,0} = (\lambda^\rho - 1)/(\rho \log \lambda)$ with probability one by the law of large numbers. Let $x$ denote the numerator: $x \equiv \sum_{i\in H_1} \lambda s_i \theta \rho /N$. Note that $i \in H_1$ is equivalent to $0 \leq s_{i,0} < 1/q$. Then, the asymptotic mean of $x$ is $x_0 = \int_0^{1/q} \lambda s_i \theta \rho \, ds_{i,0} = (\lambda^{\rho/q} - 1)/(\rho \log \lambda)$, and by the central limit theorem, $\sqrt{N}(x - x_0)$ converges in distribution to the normal distribution with mean zero and variance

$$\int_0^{1/q} (\lambda s_i \theta \rho)^2 \, ds_{i,0} - \left(\frac{\lambda^{\rho/q} - 1}{\rho \log \lambda}\right)^2 = \frac{\lambda^{2\rho/q} - 1}{2 \rho \log \lambda} - \left(\frac{\lambda^{\rho/q} - 1}{\rho \log \lambda}\right)^2. \hfill (29)$$

I regard (28) as a function $F$ of $x$. By the delta method, I obtain that $F(x)$ asymptotically follows the normal distribution with mean $F(x_0)$ and variance
\[ F'(x_0)^2 \text{Avar}(x). \] \( F(x_0) \) is calculated as

\[
\frac{\sqrt{N}}{\rho \log \lambda} \log \left( \frac{(\lambda^\rho - 1)(\lambda^{\rho/q} - 1)/(\rho \log \lambda)}{(\lambda^\rho - 1)/(\rho \log \lambda)} + 1 \right) = \frac{\sqrt{N}}{q}. \tag{30}
\]

This cancels out with the second and third terms of \( m_1/\sqrt{N} \). \( F'(x_0)^2 \text{Avar}(x) \) is calculated as \( \sigma_1^2 \) in the proposition. Then, \( m_1/\sqrt{N} \) asymptotically follows a normal distribution with mean zero and variance \( \sigma_1^2 \). This completes the proof.

## D Proof of Proposition 2

Lemma 1 implies that \( (\log K^2 - \log K_0^e)/\log \lambda \) asymptotes to \( (m_1 + M)/N \), which I focus on here. Its unconditional variance \( \text{Var}((m_1 + M)/N) \) is decomposed as follows:

\[
E \left[ \text{Var} \left( \frac{M}{N} \mid m_1 \right) \right] + \text{Var} \left( \frac{m_1}{N} + E \left[ \frac{M}{N} \mid m_1 \right] \right) \tag{31}
\]

\[
= E \left[ E \left[ \text{Var} \left( \frac{M}{N} \mid m_1, m_2 \right) \mid m_1 \right] + \text{Var} \left( E \left[ \frac{M}{N} \mid m_1, m_2 \right] \mid m_1 \right) \right] \\
+ \text{Var} \left( \frac{m_1}{N} + E \left[ E \left[ \frac{M}{N} \mid m_1, m_2 \right] \mid m_1 \right] \right).
\]

\( m_u \) asymptotically follows a martingale branching process when \( N \to \infty \) and \( \phi = 1 \). Thus, \( |M| \) conditional on \( |m_2| \) is asymptotically equivalent to the \( |m_2| \)-times convolution of \( M \) conditional on \( m_2 = 1 \). Using these facts, I obtain that

\[
\text{Var}(M/N \mid m_1, m_2) \sim |m_2| \text{Var}(M/N \mid m_2 = 1), \tag{32}
\]

\[
E[E[M/N \mid m_1, m_2] \mid m_1] \sim E[m_2 \mid m_1] E[M/N \mid m_2 = 1] \tag{33}
\]

\[
\sim m_1 E[M/N \mid m_2 = 1].
\]

Further, \( |m_2| \) conditional on \( m_1 \) asymptotically follows a Poisson distribution with mean \( |m_1| \), and the unconditional distribution of \( m_2 \) is symmetric. Since \( m_1/\sqrt{N} \) asymptotically follows \( N(0, \sigma_1^2) \) by Proposition 1, I can use
the formula $E[|m_1|/\sqrt{N}] \to \sigma_1 \sqrt{2/\pi}$. Applying these, I obtain

$$\text{Var} \left( \frac{m_1 + M}{N} \right) \sim E \left[ \frac{M}{N} \mid m_2 = 1 \right] \text{Var} \left( \frac{M}{N} \mid m_2 = 1 \right) + \text{Var}(m_2 \mid m_1)E \left[ \frac{M}{N} \mid m_2 = 1 \right]^2$$

Next, I calculate $\lim_{N \to \infty} E[M/\sqrt{N} \mid m_2 = 1]$. By Proposition 1, the probability of $M = N$ declines as $N^{-1.5}$ when $\phi = 1$. Since $N \cdot N^{-1.5}$ converges to 0, the conditional expectation of $M$ can be evaluated by using the asymptotic probability function (27) when $\phi = 1$:

$$\Pr(M = m \mid m_2 = 1) = e^{-m}m^{m-1}/m!.$$

By using inequality (Feller, 1957, p.52)

$$\sqrt{2\pi}m^{m+0.5}e^{-m+1/(12m+1)} < m! < \sqrt{2\pi}m^{m+0.5}e^{-m+1/(12m)},$$

the upper and lower bounds of the asymptotic mean of $M/\sqrt{N}$ are computed as follows:

$$\sum_{m=1}^{N} e^{-m}m^m/(m!\sqrt{N}) < \int_{0}^{N} m^{-0.5}dm/\sqrt{2\pi N} \to \sqrt{2/\pi},$$

$$\sum_{m=1}^{N} e^{-m}m^m/(m!\sqrt{N}) > \int_{1}^{N+1} e^{-1/(12m)}m^{-0.5}dm/\sqrt{2\pi N} \to \sqrt{2/\pi}.$$ 

Hence, $E[M/\sqrt{N} \mid m_2 = 1, M \leq N] \to \sqrt{2/\pi}$. Similarly, I obtain

$$E[M^2/N^{1.5} \mid m_2 = 1] \to 1/(1.5\sqrt{2\pi}).$$

Collecting the results, I obtain $\text{Var}((m+M)/N) \to (2/\pi)(\sigma_1+1/3)\sigma_1$. Hence, the capital growth rate has an asymptotic variance: $(\log \lambda)^2(2/\pi)(\sigma_1+1/3)\sigma_1$.

\textsuperscript{10}See the Technical Appendix for a detailed derivation.
E Proof of Proposition 3

Consider the case $\Gamma(K^e_0) > K^e_0$ depicted in Figure 1. $K^1$ is the fixed point of $\Gamma$ on the opposite side of $K^e_0$ from $K^2$. There exists a point between $K^1$ and $K^e_0$ at which $\Gamma$ crosses the 45 degree line from below. By applying Proposition 1, the number of adjusting firms between the point and $K^1$ follows the power law with exponent 0.5 if $\phi = 1$. Then, the tail distribution of $|\log K^1 - \log K^e_0|$ cannot decay faster than the power function with exponent 0.5.

By the selection rule, $|g^1| = \min \{ |\log K^1 - \log K^e_0|, |\log K^2 - \log K^e_0| \}$. Since the two terms in the minimization operator are independent conditional on $m_1$, I have that $\Pr(|g^1| > g \mid m_1) = \Pr(|\log K^1 - \log K^e_0| > g \mid m_1) \Pr(|\log K^2 - \log K^e_0| > g \mid m_1)$. Thus, $g^1$ conditional on $m_1$ has a tail that cannot decay faster than the power function with exponent $0.5 + 0.5 = 1$.

At the power exponent 1, the variance of $g^1$ conditional on $m_1$ decreases as $\int^N x^2 x^{-2} dx / N^2 \sim 1 / N$. Since the mean of $|m_1|$ increases as $\sqrt{N}$, proceeding as in the proof of Proposition 2, I obtain that the variance of $g^1$ decreases as $1 / \sqrt{N}$. If the tail distribution of $g^1$ conditional on $m_1$ decays more slowly than the power law with exponent 1, the variance of $g^1$ also decreases more slowly than $1 / \sqrt{N}$.

F Proof of Proposition 4

First, I show the counterpart of Lemma 1 as follows:

$$N(\log K_u - \log K_{u-1}) = \sum_{n=1}^{\infty} \sum_{i \in H_u} \left( \frac{k_{i,u-1}}{K_{u-1}} \right)^n \frac{\rho^{n-1}(\log \lambda_i)^n}{n!} + O(1/N)$$

$$= \sum_{i \in H_u} b_i^{\rho}(\sum_{n=1}^{\infty} \frac{\rho^{n-1}(\log \lambda_i)^n}{n!}) + O(1/N)$$

$$\rightarrow \sum_{i \in H_u} \log \tilde{\lambda}_i. \quad (40)$$

The right-hand side of (40) is denoted by $Z_u$. The probability for firm $j$ to be included in $H_u$ is

$$\Pr(j \in H_u | j \notin \cup_{v=2,3,\ldots,u-1} H_v) = \frac{\phi(\log K_u - \log K_{u-1}) / \log \lambda_j}{1 - \phi(\log K_{u-1} - \log K_0) / \log \lambda_j}. \quad (41)$$
This probability asymptotes to $\phi Z_u/(N \log \lambda_j)$ for large $N$. Thus, the number of firms $m_{u+1}$ that are induced to invest by $Z_u$ follows the sum of heterogeneous Bernoulli trials across $j$ with this probability.

Conditional on $m_u$, the density of drawing a particular set of firms $H_u$ is determined by the joint density of $(\lambda_i, \delta_i)$. The number of firms (a part of $m_{u+1}$) that are induced to invest by a particular firm $i \in H_u$ follows the sum of heterogeneous Bernoulli trials with probability $\phi \log \hat{\lambda}_i/(N \log \lambda_j)$. This number is asymptotically independent with the number of firms that are induced to invest by another firm $i' \in H_u$, because the two groups of firms can be regarded as being drawn from two disjoint intervals within $[0, Z_u/\log \lambda_i)$ in the support of $s_{i,u-1}$. Moreover, unconditional on the identity of $i$, the number of firms that are induced to invest by any firm in $m_u$ follows a mixture distribution of the Bernoulli trials and the distribution of $(\lambda_i, \delta_i)$. Therefore, $m_{u+1}$ conditional on $m_u$ follows a $m_u$-times convolution of that mixture distribution. This shows that $(m_u)_u$ follows a branching process.

The mean number of firms that are induced to invest by each member of $m_u$ is $\bar{\phi} = E(\log \hat{\lambda}_i) \phi E(1/\log \lambda_j)$ asymptotically. By the theorem by Otter, a cumulative sum of a branching process, with mean children $\bar{\phi}$ per parent, follows the distribution as in Proposition (Harris, 1989, p.32). This completes the proof.

### G Proof of Proposition 5

Using $s_{i,t}$, the aggregate capital can be written as $K_t = (\int a_{i,t}^{\rho/\alpha} \lambda_{i,t}^{\rho \alpha} k^{\rho} di)^{1/\rho}$. Thus, I obtain a marginal cost condition and a threshold rule as follows:

$$1 = \left( \int a_{i,t}^{\rho/(1-\rho)} \left( \frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\frac{\rho}{\alpha}} \lambda_{i,t}^{\rho \alpha} k^\rho di \right)^{\frac{1-\rho}{\rho}} B_{t-1}, \tag{42}$$

$$B_{t-1} \equiv \frac{(1 - c_1)E_{t-1}[Y_t/R_t]^{\frac{1}{\rho}}}{(1 - (1 - \delta)E_{t-1}[R_{t-1}^{-1}])E_{t-1}[(w_t/c_1)Y_t^{1-\alpha}/R_t]^{\frac{1-\alpha}{\alpha}}}, \tag{43}$$

$$k_{i,t}^* = b_{i,t} K_t, \tag{44}$$

$$b_{i,t} \equiv B_{t-1} \left( a_{i,t}^{\rho/\alpha} (\lambda_i^\rho - 1)/(\lambda_i - 1) \right)^{1/(1-\rho)}. \tag{45}$$

Using (44), the right-hand side of gap dynamics in (12) is written as a
modulo 1 of

\[
\frac{\log(1-\delta) + \log(a^0_t K_t) - \log(a^0_{t+1} K_{t+1}) + \frac{\rho}{\alpha(1-\rho)}(\log a_{i,t} - \log a_{i,t+1})}{\log \lambda_i} + s_{i,t} + 1,
\]

where

\[
a^0_t \equiv \left( \int a^{\frac{\rho}{\alpha(1-\rho)}}_{i,t} \left( \frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\frac{\rho}{1-\rho}} \lambda_i^{\rho s_{i,t}} d\lambda_i \right)^{\frac{1}{\rho}}.
\]

Then, \( s_{i,t} = (tU_t + V_t + s_{i,0} + t) \mod 1 = (tU_t + V_t + s_{i,0}) \mod 1 \), where \( U_t = (\log(1-\delta) + (\log(a^0_t K_0) - \log(a^0_t K_1))/\log \lambda_i \) and \( V_t = (\rho/(\alpha(1-\rho)))((\log a_{i,0} - \log a_{i,t})/\log \lambda_i \). Since \( a^0_t K_t \) converges to the steady state, \( U_t \) asymptotically follows \( U = \log(1-\delta)/\log \lambda_i \) which has a well-defined density. Since \( a_{i,t} \) is an i.i.d., bounded random variable, \( V_t \) also has a well-defined density that is common for any \( t \). Now, \( tU \) taken modulo 1 converges in distribution to a unit uniform random variable as \( t \to \infty \), and its sum with an absolutely continuous random variable, taken modulo 1, also converges to the unit uniform distribution (Engel, 1992, pp.28-29). The sum of a unit uniform random variable and an independent random variable \( s_{i,0} \), taken modulo 1, also follows the unit uniform distribution. This completes the proof.

**H Details on calibration and computation**

Firms’ markup rate \( 1/(\eta - 1) \) is set at 10%. The capital intensity \( \alpha \) is set such that the labor share \( \bar{w}\bar{L}/\bar{Y} \) is equal to 0.67. The annual rate of depreciation is set at 8% and the annual risk free rate at 4%. The disutility from labor is specified as a quadratic function. Indivisibility parameter \( \lambda_i \) is a random variable drawn in period 0 and fixed for later periods. I set that \( \lambda_i \) is drawn from a normal distribution with mean 1.028 and standard deviation 0.004 truncated at two standard deviations. I choose this specification to match with the 2.8% plant Herfindahl index estimated by Ellison and Glaeser (1997). Plant Herfindahl measures the representative share of a plant’s employment in an industry. When capital size is adjusted by changing the number of plants, plant Herfindahl can be interpreted as a lower bound of capital indivisibility, which coincides with firm-level capital indivisibility if the industry is a monopoly. These parameters and steady state values for the benchmark specification are summarized in Table 1.
The number of firms $N$ is set at 350000 to match with the number of operating manufacturing plants in the US (Cooper, Haltiwanger, and Power (1999)). The logarithm of the idiosyncratic productivity $\log a_{i,t}$ is assumed to follow a normal distribution with standard deviation 0.05%. The mean productivity is set such that the mean of $\bar{a}_{i,t}^{\rho/(\alpha(1-\rho))}$ (which appears in the threshold rule (44)) is normalized to 1. In the initial period, $s_{i,0}$ is randomly drawn from a uniform distribution, and in each period, productivity $a_{i,t}$ is drawn independently. The equilibrium path is simulated for 1100 periods, from which the first 100 periods are discarded. The reported moments in Table 2 are averages of 10 simulated runs. Figures in parentheses are standard errors.

I Information content of $s_{i,t}$

In this section, I examine whether the exact gap distribution $s_{i,t}$ has prediction power for future prices in the simulated model. The exact gap distribution $s_{i,t}$ enters the equilibrium system (19, 20, 21) in summations directly and in information sets of conditional expectations. The former effect is negligible. The difference between the summations in (21) when evaluated by exact gaps and by the integral over the stationary distribution is less than $10^{-13}$ percent in the benchmark simulation. As $N$ becomes larger, this error becomes even smaller. The latter effect by the gap distribution in information sets might be more potent. However, fully implementing this state variable raises the curse of dimensionality in numerical computation. Krusell and Smith, Jr. (1998) deal with this by transforming the distribution equivalently to an infinite vector of moments and then approximating it by a finite vector. In this model, I approximate the gap distribution by its stationary counterpart in a continuum economy. In what follows, I check that the gap distribution does not have significantly higher prediction power for future prices than the stationary distribution.

For each period $t$, the expectation for investment threshold $s_{i,t}^*$ is formed as (17). The expected investment is the integral of indivisible investments of firms with $s_{i,t}$ less than $s_{i,t}^*$. Actual investment may differ from the expected value due to two factors: the exact values of $s_{i,t}$ and the realizations of idiosyncratic productivities. If the exact $s_{i,t}$ is known for some future period $t$ and no productivity shocks are present, then the distribution of $s_{i,t}$ should have prediction power for the difference between the expected and
realized investments. This prediction power is weakened when the idiosyncratic shocks wash out the information $s_{i,t}$ has. To see this, I conduct the following experiment. In a variation of the benchmark simulation, I compute the equilibrium path selected by mechanism 2. I additionally compute $\log(\Gamma(K_{t+1}^e) - \log K_{t+1}^e)$ for the case where productivity shocks in $t+1$ are set at 0. This log-difference and $\log K_{t+1} - \log K_{t+1}^e$ turn out to exhibit no correlation either in terms of values or signs. This implies that exact $s_{i,t}$ alone cannot even predict the sign of expectation error because of the uncertain productivities. This experiment confirms that the exact gap distribution has little prediction power for future prices even at the small standard deviation (0.05%) of idiosyncratic productivity shocks, because aggregate investments respond to the small perturbations in productivity or capital gap non-linearly and quite sensitively.

References


