The Taylor Principle and the Taylor Rule Determinacy Condition in the Baseline New Keynesian Model: Two Different Kettles of Fish

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Abstract
The Taylor Principle (1993) suggests monetary policy should make the interest rate move in the same direction and by a greater amount than observed movements in inflation. The resulting co-movements between inflation and the real interest, as Taylor (1999) demonstrated for a simple, “backward-looking” model, can be shown to be necessary for stability in models of that type. In the context of the baseline New Keynesian, “forward-looking” model, (NKM) as exposited by Woodford (2001, 2003) a related “stability and uniqueness” (or determinacy) condition arises which is necessary for a determinate solution. Woodford and many others have interpreted this condition as representing the Taylor Principle condition.

In this paper we argue that (1) the dynamic interpretation Woodford and others put on the NKM determinacy condition is inappropriate; (2) the standard NKM determinacy condition does not even require (in general) the Taylor Principle (i.e., real interest rate moving together with inflation) to hold. The Taylor Principle and the NKM determinacy condition are two different kettles of fish.
I. Introduction

In recent years a great deal of attention among monetary economists has been focused on the issue of uniqueness, stability and/or “determinacy” in macroeconomic models, particular in the New Keynesian Model (NKM). Modeling in NKM usually represents monetary policy as following a Taylor rule, and the parameters of the Taylor rule must meet certain “determinacy conditions” which may be necessary to rule out both “sunspots” and explosive solutions. At least since the widely-cited article by Woodford (2001) there has been a consensus that the determinacy condition in these models is essentially a restatement of the Taylor Principle – that the policy rule must guarantee that the real interest rate will move together with inflation. This movement of real interest rates contains demand and inflation responses to shocks that would otherwise create explosive or stable sunspot solutions. We beg to differ with this line of reasoning, at least as applied to the NKM.

Our argument is as follows. “Backward-looking” models (such as that by Taylor (1999) himself) do require the Taylor Principle to hold for “stability” - meaning here where projected
paths eventually approach long-run or steady-state solutions of the endogenous variables. But one should avoid conflating the Taylor Principle with the determinacy condition in the forward-looking NKM. The determinacy condition in NKM, when met, assures us that the model’s solution is unique. The dynamics are “stable” in the above sense by construction (all shocks are AR(1)). If the determinacy condition is not met, the solution is “immediately explosive” (no finite solution exists) or may result in non-explosive sunspot solutions.¹

To illustrate our point, we begin by showing how the two concepts apply in simple, univariate backward- and forward-looking models. Next we turn to two representative bivariate (inflation and output gap) backward- and forward-looking models. The backward-looking model is Taylor’s (1999) model; the forward-looking model is the baseline standard NKM model as exposited by Clarida, Gali and Gertler (1999), Woodford (2001, 2003) and others. Finally, we present our most decisive evidence against the view that determinacy requires the Taylor Principle in NKM models: we provide an example of a NKM-Taylor rule model which is determinant but the real interest rate moves in the opposite direction of inflation.
II. A Univariate Backward-Looking Model

A basic univariate backward-looking model can be written:

\[ x_t = a + b x_{t-1} + u_t, \text{ where } u_t = \rho u_{t-1} + \eta_t \text{ and } 0 < \rho < 1. \eta_t \text{ is a white noise.} \]

Iterating the model forward, we rewrite (1) as:

\[ \tilde{x}_{t+n} = (1 + b + b^2 + \cdots + b^n) a + b^{n+1} \tilde{x}_{t-1} + \sum_{i=0}^{n} \rho^i b^{n-i} u_t \]

where \( n \) is the number of future periods. From equation (2) we see that the dynamics of process \( \{u_t\} \) determine the future path of \( \tilde{x}_t \). Figure 1 illustrates the effects of a positive unit shock in \( u_t \) on \( \tilde{x}_t \). These paths involve well-defined equilibria for different times. We consider only positive values of \( b \). Whether \( b \) is greater or less than one, the solutions are determinate. The value of \( b \) does determine whether the path is a convergent path or a divergent path. As \( n \) approaches with \( b < 1 \), the solution path exists and it converges to a certain value. As \( n \) goes to infinity with \( b \geq 1 \), the solution path still exists but also approaches infinity, i.e., \( \tilde{x}_{t+n} = \infty \). The problem is, for the cases which \( b \geq 1 \), the path strays indefinitely from the long-run equilibrium. This is “instability” in the backward-looking case.
III. A Univariate Forward-Looking Model

Contrast this with a forward-looking univariate model such as:

(3) \[ \tilde{x}_t = a + bE_t[\tilde{x}_{t+1}] + u_t \]

where \( u_t = \rho u_{t-1} + \eta_t \) and \( 0 < \rho < 1 \). \( \eta_t \) is a white noise. Again using the iteration method, we rewrite equation (3) as:

(4) \[ \tilde{x}_t = (1 + b + b^2 + \cdots + b^n) a + b^nE_t[\tilde{x}_{t+n}] + (1 + b \rho + b^2 \rho^2 + \cdots b^n \rho^n)u_t. \]

The necessary condition for convergence in equation (4) is \( b < 1 \). \( b < 1 \) guarantees that the
first and second terms of the right hand side of equation (4) have defined limits. If sunspots are present, they will drop out of the solution. Given the stationary process \{u_t\}, the coefficient of \(u\) has a limit as \(t\) goes to infinity. Given this condition, the value of solution paths will follow

\[
(5) \quad \tilde{x}_t = \frac{a}{1-b} + \frac{1}{1-b\rho} u_t,
\]

and convergence to this path is immediate. If \(b > 1\) and there are no sunspots, the model explodes instantaneously. If \(b > 1\) and there are sunspots, there may be non-explosive but arbitrary solutions. Either way, we say the solution is “indeterminate.”

Different values of \(b\) (<1) in equation (5) imply different solution paths. Figure 2 illustrates the dynamics for each unique path with a unit shock in \(u_t\) on \(\tilde{x}_t\). Where a determinate solution exists, the path is stable (tendency to return to the long-run solution) by construction.
Note: Figure 2 illustrates the existence of solution paths of $\tilde{x}_t$. When $b > 1$, the solution paths are indeterminate. When $b < 1$, the solution paths are determinate; the pattern of the paths depends on the magnitude of $b$. As the figure shows, each $b$ has an unique path.
IV. **Taylor (1999)**

Now consider the bivariate (inflation and output) counterparts to univariate models. We start

with the bivariate backward-looking model by Taylor (1999):

\[
\begin{align*}
(6) \quad x_t &= -\beta (i_t - \pi_t - r) + g_t \\
(7) \quad \pi_t &= \pi_{t-1} + \alpha x_{t-1} + u_t \\
(8) \quad i_t &= \varphi_0 + \varphi_\pi \pi_t + \varphi_x x_t
\end{align*}
\]

where \( x \) = output gap; \( \pi \) = inflation rate; \( i \) = short term nominal interest rate. \( g \) and \( u \) are (independent and not auto-correlated) shocks with zero mean. The model parameters \( \alpha, \beta \) are positive. \( r \) is the natural rate of interest. \( \varphi_0, \varphi_\pi \) and \( \varphi_x \) are the policy parameters of the Taylor rule.

The solution path of inflation can be written as:

\[
\begin{align*}
(9) \quad \pi_{t+i} &= \sum_{i=1}^{n} \Lambda^{i-1} g_{t+i} + \frac{\alpha}{1 + \beta \varphi_x} \sum_{i=1}^{n} \Lambda^{i-1} u_{t+i} + \Lambda^{i-1} \pi_t + \alpha \Lambda^{i-1} x_t \\
\end{align*}
\]

where \( \Lambda = \frac{\alpha \beta (1 - \varphi_\pi) + (1 + \beta \varphi_x)}{1 + \beta \varphi_x} \). The key parameter is \( \Lambda \), which if less than unity will indicate

that projections of \( \pi_{t+n} \) will approach the model’s steady-state value \( \frac{\varphi_0 - r}{1 - \varphi_\pi} \) after shocks in \( u \) or \( g \) – hence, the model is “stable” in the sense Taylor intended. If \( \Lambda > 1 \), the model’s projections will depart continuously from steady-state values – “unstable” in the Taylor sense, though not “explosive” in the immediate sense. Each period’s projection is well-defined.

Since \( \Lambda = 1 + \frac{\alpha \beta (1 - \varphi_\pi)}{1 + \beta \varphi_x} \), the Taylor “stability” condition can be simplified to \( \varphi_\pi > 1 \).\(^5\) This establishes the Taylor Principle and its role in this model.
V. Clarida, Gali and Gertler (1999)

Now while the Taylor Principle governs whether a backward model like Taylor’s above will be “stable” in terms of tending toward steady-state, there is no similar implication for forward looking models. To illustrate this point, we turn to the bi-variate forward-looking model of Clarida, Gali and Gertler (1999, henceforth CGG (1999)). Their baseline NKM model consists of the familiar equations for the output gap (“IS”) and inflation (“Phillips Curve”):

\begin{align}
 x_t &= E_t x_{t+1} - \frac{i_t - E_t \pi_{t+1}}{\sigma} + g_t \\
 \pi_t &= \beta E_t \pi_{t+1} + u_t
\end{align}

where \( x \) represents output gap in logs; \( \pi \) represents inflation (log-deviation from steady-state); \( i \) represents the nominal interest rate (deviation from steady-state); \( 0<\beta<1 \) is a discount factor, \( \kappa > 0 \) is the Phillips Curve parameter reflecting the degree of price flexibility (higher means more), and \( \sigma>0 \) is the consumption-elasticity of utility. \( g_t \) and \( u_t \) are shocks of AR(1) form: \( u_t = \rho u_{t-1} + \eta_t \) (\( 0<\rho<1 \)) and \( g_t = \lambda g_{t-1} + \varepsilon_t \) (\( 0<\lambda<1 \)).

The solutions for the optimal path of CGG (1999) under discretion (time-consistent) as:

\begin{align}
 \pi_t &= \frac{\Gamma u_t}{\kappa^2 + \Gamma (1 - \beta \rho)} \\
 x_t &= -\frac{\kappa u_t}{\kappa^2 + \Gamma (1 - \beta \rho)}
\end{align}

where \( \Gamma \) is the weight on the output gap in the welfare function.
Three observations should be made about these paths: first, the IS or prospective productivity shock $g_t$ does not appear. Since these “demand-side” shocks move $\pi_t$ and $x_t$ in the same direction, an optimal path requires no response of these variables to $g_t$ shocks.

Second, these paths are by construction “stable” (i.e., in the sense the variables approach steady-state solutions) on account of the specification of the $u$ shock.

Third, these paths can be (were) derived without specification of any monetary policy rule. To learn whether paths (12) and (13) will obtain it is essential to specify the monetary policy rule and its determinacy conditions.

VI. Optimality, Determinacy, and the Taylor rule

A variety of monetary policy rules can meet the requirements of (12) and (13), but the Taylor rule has been universally employed. We write the particular Taylor rule as

\begin{equation}
    i_t = \varphi_x \pi_t + \varphi_x x_t + \varphi_g g_t
\end{equation}

which includes no intercept (the model is derived as log-linearized deviations from steady state) and contains a $\varphi_g$ parameter to offset $g$ shocks.\footnote{Putting (14) and (15) into (10 and (11), the model can be written in the form

$$M_t = AM_{t+1} + e_t$$

where $M'_t = [x_t \pi_t], M'_{t+1} = [E_t[x_{t+1}] E_t[\pi_{t+1}]],$

$A$ is a two-by-two coefficient matrix and $e_t$ is a vector of exogenous shock terms. The solution of the model will be linear in $u_t$ and $g_t$. Bullard and Mitra (2002) showed that}
uniqueness/stability or determinacy of this model requires that the eigenvalues of A lie inside the unit circle, which I turn then requires

\begin{equation}
0 < (1 - \beta) \varphi_x + \kappa (\varphi_\pi - 1).
\end{equation}

This is the condition that rules out explosive solutions as well as “sunspot” solutions.

In this case the optimal Taylor rule is

\begin{equation}
\varphi_g = \sigma
\end{equation}

\begin{equation}
\varphi_\pi = \rho + \frac{\kappa \sigma (1 - \rho)}{\Gamma} + \frac{\kappa \varphi_x}{\Gamma}
\end{equation}

which is obviously not unique in \(\varphi_\pi\) and \(\varphi_x\). As illustrated in Figure 3, the unique, optimal paths (10) and (11) will necessarily be reached for all combinations of \(\varphi_\pi\) and \(\varphi_x\) that lie to the Northeast of the borderline for determinacy (15).

An interesting implication of the optimal paths created by this optimal Taylor rule - (16) and (17) – is that, since all periods require \(x/\pi = -\kappa/\Gamma\) the model can be written in the form of two, univariate equations (\(x_t\) as a function of \(E_{x_{t+1}}\) and \(\pi_t\) as a function of \(E_{\pi_{t+1}}\)) along the lines of our univariate, forward-looking example in Section III. In this “optimized” solution path, the \(b<1\) condition noted in the earlier example provides a less “stringent” condition for determinancy than (15). The point on the optimal \(\{\varphi_\pi, \varphi_x\}\) locus in Figure 3 is marked as \(\varphi''\). (More details in the Appendix.)
VII. Determinate Solutions Without the Taylor Principle

With this background and argument of the previous sections we now return to the classic interpretation of the NKM-Taylor rule determinacy condition by Woodford (2001, p. 233), probably the most elegantly expressed and most widely cited:

The determinacy condition...has a simple interpretation. A feedback rule satisfies the Taylor Principle if it implies that in the event of a sustained increase in the inflation rate by k percent, the nominal interest rate will eventually be raised by more than k
percent. In the context of the model sketched above, each percentage point of permanent increase in the inflation rate implies an increase in the long-run average output gap of $\frac{1-\beta}{k}$ percent; thus a rule of the form conforms to the Taylor Principle if and only if the coefficients $\varphi_\pi$ and $\varphi_x$ satisfy $\varphi_\pi + \frac{1-\beta}{k} \varphi_x > 1$. In particular, the coefficient values necessarily satisfy the criterion, regardless of the size of $\beta$ and $k$. Thus the kind of feedback prescribed in the Taylor rule [$\varphi_x = 0.5, \varphi_\pi = 1.5$] suffices to determine an equilibrium price level.

What is our objection to the statement above? First, the paragraph implies that the determinacy condition influences the dynamic paths of $\pi_t$ (and presumably $x_t$), whereas we noted earlier that (15) ensures unique paths for $\pi$ and $x$ which are reached instantly. Second, Woodford’s statement suggests that in order for these unique paths to obtain the Taylor Principle must be apply – i.e., the real interest rate must rise with inflation. It has become nearly universal to assert that positive co-variation between the real interest rate and inflations critical in avoiding explosive solutions and sunspots.

It turns out this second suggestion is fairly easy to refute. The standard determinacy condition (15) does not require the real interest rate to co-vary positively with inflation. We assume again that $g_t$ is offset by the setting (16) ($\varphi_g = \sigma$). We will however (for reasons we will explain in a moment) apply the optimal settings shown in (17). The solutions for $\pi_t$ and $x_t$ will be proportional to $u_t$, which means that the expected $t+1$ values of these variables can be
written as \( \rho \) times their current values. Then differentiating (10) with \( i_t \), represented by the Taylor rule and (11), one arrives as a constraint between movements in the real interest rate \( (r_t) \), which can also be written

\[
r_t = \left( \frac{\phi_{\pi}}{\rho} - 1 \right) E_t(\pi_{t+1}) + \phi_x x_t = \left( \frac{\phi_{\pi}}{\rho} - 1 \right) \rho \pi_t + \phi_x x_t
\]

and \( \pi_t \) as

\[
\begin{align*}
\frac{d r_t}{d \pi_t} &= \left( \frac{\phi_{\pi}}{\rho} - 1 + \frac{\phi_x (\rho - \phi_x)}{\sigma \left( 1 - \rho + \frac{\phi_x}{\sigma} \right)} \right) d(\pi_t) \\
(18)
\end{align*}
\]

The term in the brackets of (18) must be negative if \( \phi_{\pi} < \rho \) for any positive \( \phi_x \), and \( \phi_x > 0 \) is required by the determinacy condition (15). For any policy setting that is not explosive or accommodative of sunspots, the real rate must decline in inflation when \( \phi_{\pi} < \rho \), rise in inflation when \( \phi_x > \rho \), or indeed not change in inflation (interest rate change matches inflation) when \( \rho = \phi_{\pi} \). Figure 4 presents an illustrative simulation of the period \( t \) effects of \( u_t \) on the real interest rate for various settings of \( \phi_{\pi} \).
We use this example to make a point – that determinate models do not require the real rate to move together with inflation - , and not to suggest that having $\varphi_\pi < \rho$ would ever be good policy. In the context of this baseline model, for instance, it is easy to show that such a setting
must be sub-optimal. Figure 5 illustrates the effect of constraining $\phi_\pi$ to be equal or below $\rho$.

Determinate Taylor rule settings are restricted to a range that cannot include the optimal setting.

**Figure 5. When $\phi_\pi$ is Constrained to be $\leq \rho$**

![Diagram](image)

Note: The solid ray denotes the optimal condition $\phi_\pi = \rho + \frac{k(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \phi_x$. The dotted line indicates the boundaries of the determinacy condition $k(\phi_\pi - 1) + \phi_x(1 - \beta) > 0$. The intercept of solid ray and the vertical axis must be greater than $\rho$.

That the optimal setting of the Taylor rule must involve the Taylor Principle restates the finding of CGG (1999, p. 1672) for this model in which the optimal interest rate is (using our notation)

\begin{equation}
(19) \quad i_t = \left[ 1 + \frac{(1-\rho)k \sigma}{\rho \Gamma} \right] E_t \pi_{t+1} \quad i_t = \left[ 1 + \frac{(1-\rho)k \sigma}{\rho \Gamma} \right] \rho \pi_t
\end{equation}
where the term in brackets must be greater than unity. To say that the Taylor Principle should apply is of course using “should” in the normative sense.

VIII. **Summary**

The literature has conflated the Taylor Principle with the NKM determinacy condition.

They are two different kettles of fish.
Appendix

To demonstrate the optimized, time-consistency model of CGG (1999) has a less “stringent” determinacy condition than the standard determinacy condition: First, solve the model for $\pi_t$ and $x_t$ as function of their two $t+1$ values. Then (a) convert the model into a two-equation, univariate model using the first order condition, $x_t = -\frac{k}{\Gamma} \pi_t$; also (b) constrain the $\varphi$’s to follow the optimal condition:

\begin{equation}
(\varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \varphi_x).
\end{equation}

Find the value of $\varphi_x$ that makes the coefficient on the $t+1$ value of $x$ equal to 1.0. This is the borderline condition for determinacy in the univariate model. This value of (call it $\varphi_x'$) equals

\begin{equation}
\varphi_x' = \frac{(\rho-1)k\Gamma+(2-\rho)k^2\sigma+(1-\beta)\sigma\Gamma}{-k^2-\Gamma+\beta\Gamma}.
\end{equation}

This is “borderline” $\varphi_x'$ which, along with the associated $\varphi_\pi$, will meet optimality/time-consistency and make the coefficient on the future value equals to one.

Second, find the intersection of $\varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \varphi_x$ and $k(\varphi_\pi - 1) + (1 - \beta)\varphi_x = 0$. This is the optimal $\varphi$-set at just the point that satisfies the standard determinacy condition (15). The solution of the intersection is

\begin{equation}
\varphi_x'' = \frac{(\rho-1)(k\Gamma-k^2\sigma)}{-k-\Gamma+\beta\Gamma}.
\end{equation}

This is the “borderline” $\varphi_x''$ (with the associated $\varphi_\pi$) that is optimal and meets the standard
determinacy condition.

Third, subtract (A2) from (A1):

\[(A3) \quad \varphi_x'' - \varphi_x' = \frac{-k^2 \sigma - (1-\beta) \sigma \Gamma}{-k^2 - (1-\beta) \Gamma} \cdot \]

The value of (A3) is positive given the fact that both numerator and denominator are negative. That is, $\varphi_x'$ lies inside the standard determinacy region in Figure 3.
REFERENCES


This paper does not analyze “sunspot” solutions. These are rational solutions that reflect certain arbitrary and “non-fundamental” disturbances to expectations which produce self-fulfilling and non-explosive solutions when mathematical determinacy conditions are not met. When the determinacy conditions are met, the effect is to rule out (“cancel out”) the effects of the sunspots. Our focus will be on whether meeting the determinacy conditions has dynamic implications apart from ruling out sunspots and/or being instantly explosive. We will argue that determinacy conditions do not have such implications for the baseline New Keynesian Model, notwithstanding widespread claims they do.

When \( n = 0 \), equation (2) is the current period. Suppose \( n = 5 \), equation (2) expresses that current value of \( x \) is the sum of 5 periods forward, i.e., \( \tilde{x}_{t+5} = (1 + b + b^2 + b^3 + b^4 + b^5)\alpha + b^6\tilde{x}_{t-1} + \sum_{i=0}^{5} \rho^i b^{5-i} u_t \), where \( \sum_{i=0}^{5} \rho^i b^{5-i} = b^5 + b^4 \rho + b^3 \rho^2 + b^2 \rho^3 + b \rho^4 + \rho^5 \).

When \( n \to \infty \), given \( b < 1 \), equation (2) converges to \( \tilde{x}_{t+\infty} = \frac{a}{1-b} \).

\[ E_t[u_{t+1}] = b\rho u_t, E_{t+1}[u_{t+2}] = b^2\rho^2 u_t \text{ and } E_{t+n-1}[u_{t+n}] = b^n\rho^n u_t. \]

For \( \Lambda < 1 \), \( \frac{\alpha \beta (1-\phi_n)}{1+\beta \phi_x} \) must be negative. Given positive \( \alpha, \beta \) and \( \phi_x \), this implies \((1 - \phi_n) < 0\).

Woodford (2001) proposes this as a necessary parameter to reach optimality in the baseline NKM with Taylor rule.

Assumed parameter values are \( \rho=.5, \Gamma=2, \kappa=.3, \text{ and } \sigma=1. \)

This is the f.o.c. of the time-consistent “discretionary” solutions of CGG(1999), page 1672.

We emphasize that this is not the only set of assumptions that can lead to the real interest rate moving in the opposite direction of inflation. For example, Thurston (2012) notes conditions in this model that produce a constant nominal interest rate and a negative co-movement of the real interest rate and inflation.