The Value of Uncertainty under Limited Commitment

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Abstract

In this paper, I analyze a simple financial contracting problem between a risk-neutral financial intermediary and a risk-averse household, who receives a potentially stochastic and growing endowment and is unable to commit to the contract. I argue that in this environment, the presence of uncertainty in the household's endowment may improve the household's welfare by relaxing its commitment problem. I analytically and numerically examine the conditions under which this is the case, and show that it requires sufficiently rapid growth of the household's endowment. This result suggests that growth slowdown may have a disproportionately large effect on welfare, which has potentially important practical implications, for example, for developing countries.

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1 Introduction

Uncertainty is typically a source of welfare loss in economics. In a standard Walrasian economy, agents may be able to insure in the market some or all of the uncertainty they face, but they can at best be as well as off as under the no-uncertainty case. There are some classes of models, however, in which the presence of uncertainty may improve welfare relative to the certainty case. Among them is a model of dynamic contract with limited commitment, which I discuss in this paper. In this type of model, feasible contracts must satisfy the participation constraints of the agents who cannot commit to the contract, typically generating efficiency loss compared to the full commitment case. The presence of uncertainty may lower the outside option values of such agents, which relaxes their participation constraints and expands the set of feasible contracts, and hence potentially improves the welfare of the parties involved. There has been little work, however, that formally pursues this issue.

In this paper, I analyze a simple financial contracting problem between a risk-neutral financial intermediary and a risk-averse household, who receives a potentially stochastic and growing endowment and is unable to commit to the contract. I analytically and numerically examine the conditions under which the presence of uncertainty in the household's endowment improves its welfare compared to the no-uncertainty case, and show that it requires sufficiently rapid growth of endowment. This result suggests that growth slowdown may have a disproportionately large effect on welfare, which has potentially important practical implications, for example, for developing countries, if we reinterpret the household as a small open economy and the financial intermediary as an international creditor[1].

The model of this paper is a variant of the "villager-moneylender" model described in Ljungqvist and Sargent (2004), which analyzes the optimal loan contract under one-sided lack of commitment. The biggest difference is that the endowment of the villager grows over time in my model, which creates original welfare implications as I describe below. Krueger and Perri (2006) consider a risk-sharing problem between two ex-ante identical households, and show that a larger income risk may enable better risk-sharing by reducing their autarky values. While this mechanism is essentially identical to the one I resort to, in their environment with no trend income growth, the first best outcome is the perfect risk sharing, which contrasts with mine in which the presence of uncertainty may strictly improve the agent's welfare compared to the no-uncertainty case. The key to this difference is that a contract has two potential roles in my model. The first is efficient risk sharing,

[1] Kehoe and Perri (2002) consider a two-country model in which the countries are unable to commit to the repayment of international loans, and examine the international business cycle properties in such an environment.

or consumption smoothing across different states of nature, which is the center of attention in typical optimal contract literature. If this is the only role performed by the contract, obviously the allocation under no uncertainty is the first best, and hence uncertainty can not improve welfare even if it relaxes the participation constraints. Incorporating growth into the model, however, creates the second role: intertemporal consumption smoothing, or efficient consumption allocation over time. In such an environment, uncertainty may improve welfare over the non-uncertainty case as it helps achieve better intertemporal consumption smoothing by reducing the incentives for the agents to walk away from the contract.

There is an extensive literature on the limited commitment environment. To name some of the important contributions, Thomas and Worrall (1988) and Kocherlakota (1996) develop the optimal contract approach to this environment, which I adopt in this paper. The competitive equilibrium approach, which attempts to achieve the constrained efficient allocations in a decentralized setup, is developed in Kehoe and Levine (1993) and Alvarez and Jermann (2000).

2 Model

2.1 Basic Environment

Time is continuous[2] and goes to infinity, and there is a single perishable good in the economy. There are two types of agents. Households are infinitely lived, have a CRRA period utility over consumption, and discount future with the discount rate $\rho > 0$, so their lifetime utility is given by

$$U = \int_0^\infty e^{-\rho t} u(C(t)) dt,$$

where $u(C(t)) = C(t)^{1-\sigma}/(1-\sigma)$ ($\sigma > 0$, $\sigma \neq 1$). Financial intermediaries are risk-neutral, and unlike households, have access to external financial market where they can borrow or lend at a constant interest rate $r > 0$.

At date t, a household receives a stochastic endowment $Y(t, s(t))$, which grows over time

[2]None of the results in the paper hinges on the assumption of continous time, except that exact expressions for the household's welfare may only be derived under continous time, which allows computing the exact time remaining until the participation constraint starts to bind when the state remains unchanged. This issue does not arise in typical literature in which the growth rate of endowment and of optimal consumption both equal zero.

at a constant rate $g \geq 0$. That is,

$$\begin{aligned} Y(t, s(t)) &= e^{gt}\beta \text{ if } s(t) = H \text{ (high state)} \\ &= e^{gt}(1-\beta) \text{ if } s(t) = L \text{ (low state)} \end{aligned}$$

, where $\beta \in [\frac{1}{2}, 1]$. The state $s(t) = \{H, L\}$ switches from one state to the other with a Poisson arrival rate λ, and the probability of being in each state at $t = 0$ equals the long-run probability, $\frac{1}{2}$. This is the only source of uncertainty in the model[3].

A household and a financial intermediary engage in a long-term loan contract before the realization of the state at $t = 0$: the household promises to offer its endowment stream to the financial intermediary, and receives in return a consumption stream. By assumption, the financial intermediary can commit to the contract, but the household cannot. The household can walk away from the contract at any moment, but doing so results in a permanent exclusion from the loan market. The loan market is assumed to be competitive. As financial intermediaries are risk neutral, they make zero profits in the equilibrium.

In the following, I assume that

$$r > g \tag{1}$$

and

$$r(1-\sigma) < \rho \tag{2}$$

are satisfied. Since $\rho > 0$ and $\sigma > 0$, these assumptions also imply

$$g(1-\sigma) < \rho. \tag{3}$$

These conditions guarantee that the optimization problems in the model are well-defined.

2.2 Optimal Contract

In the rest of the paper, an "optimal contract" implies a feasible contract that maximizes the financial intermediary's profit, for a given initial promised value, or equivalently the expected present value of lifetime utility, to the household. When the resulting profit equals zero, as will be the case in equilibrium, it is called an "optimal zero profit contract".

The optimal contract is a stream of consumption $\{C(s^t)\}, t = [0, \infty)$ that solves, for a

[3] At least in this verion of the paper, I only analyze the problem between a single household and financial intermediary. So the cross-sectional distribution of endowment is irrelevant.

given V,

$$\Pi = \max_{C(s^t)\geq 0,\ \forall s^t} E_0 \int_0^\infty e^{-rt}\{Y(t,s(t)) - C(s^t)\}dt$$
$$s.t. \quad E_0 \int_0^\infty e^{-\rho t} u\{C(s^t)\}dt \geq V$$
$$E_t[\int_t^\infty e^{-\rho(\tau-t)} u\{C(s^\tau|s^t)\}d\tau|s^t] \geq V^{Aut}(t,s(t)) \quad \forall s^t$$

where s^t denotes the history $s(\tau) = \{H, L\}, \tau = [0,t)$, and $\{C(s^\tau|s^t)\}, \tau = [t,\infty)$ denotes the consumption stream specified by $\{C(s^t)\}$, following the history s^t. $V^{Aut}(t,s(t))$ is the autarky value for the household, given by

$$V^{Aut}(t,s(t)) = E_t[\int_t^\infty e^{-\rho(\tau-t)} u\{Y(\tau,s(\tau))\}d\tau|s(t)].$$

Note that due to the assumptions above on the process of $Y(t,s(t))$ and $s(t)$, $V^{Aut}(t,s(t))$ depends only on $s(t)$, the current state at t, and not on the entire history s^t.

2.3 Recursive Optimal Contract

In order to examine the path of consumption in the optimal contract, it is convenient to consider a recursive formulation of the problem. Let us define $\gamma \equiv \rho - (1-\sigma)g$. From (3), $\gamma > 0$. Then the financial intermediary's recursive problem is,

$$\max_{c,v'(s')} \pi(v,s) = \frac{1}{1+(r-g)\Delta}\{[y(s)-c]\Delta + (1-\lambda\Delta)\pi(v'(s),s) + \lambda\Delta\pi(v'(-s),-s) + o(\Delta)\} \tag{4}$$

$$v = \frac{1}{1+\gamma\Delta}\{u(c)\Delta + (1-\lambda\Delta)v'(s) + \lambda\Delta v'(-s)\} \tag{5}$$

$$v'(s') \geq v^{Aut}(s'), \quad s' = \{s, -s\} \tag{6}$$

, where $-s = H$ if $s = L$, and vice versa. Δ is a very short period of time which will eventually be taken to zero, and $o(\Delta)$ is an approximation error term such that $\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$. The lower case letters are obtained by detrending $Y(t,s(t))$ and $C(s^t)$ with e^{gt}, and the utility related terms with $e^{g(1-\sigma)t}$: for example, $y(s) = e^{-gt}Y(t,s) = Y(0,s)$ and $v^{Aut}(s) = e^{-g(1-\sigma)t}V^{Aut}(t,s) = V^{Aut}(0,s)$.

The detrended autarky values in the high and low state, $v^{Aut}(H)$ and $v^{Aut}(L)$, can be computed as

$$v^{Aut}(H) = \int_0^\infty e^{-(\lambda+\rho)t} e^{g(1-\sigma)t}\{u(\beta) + \lambda v^{Aut}(L)\}dt$$
$$v^{Aut}(L) = \int_0^\infty e^{-(\lambda+\rho)t} e^{g(1-\sigma)t}\{u(1-\beta) + \lambda v^{Aut}(H)\}dt$$

This leads to

$$v^{Aut}(H) = \frac{(\lambda+\gamma)u(\beta)+\lambda u(1-\beta)}{\gamma(2\lambda+\gamma)},\ v^{Aut}(L) = \frac{(\lambda+\gamma)u(1-\beta)+\lambda u(\beta)}{\gamma(2\lambda+\gamma)} \tag{7}$$

or alternatively,

$$u(\beta) = (\lambda+\gamma)v^{Aut}(H) - \lambda v^{Aut}L \tag{8}$$

$$u(1-\beta) = (\lambda+\gamma)v^{Aut}(L) - \lambda v^{Aut}(H) \tag{9}$$

2.4 Consumption Path

Let θ_1 and $\theta_2(s')(s' = \{s, -s\})$ respectively denote the Lagrange multipliers for (5) and (6). Setting up the Lagrangian, taking the first order conditions as well as the envelope condition and combining them, it follows that

$$\theta_2(s') = \frac{1}{1+(r-g)\Delta}\{\frac{1+\gamma\Delta}{1+(r-g)\Delta}\frac{1}{u'(c(s'))} - \frac{1}{u'(c)}\}.$$

This implies that, if the participation constraint doesn't bind, $\theta_2(s') = 0$ so that

$$\frac{u'(c)}{u'(c(s'))} = \frac{(c(s'))^\sigma}{c^\sigma} = 1 + \frac{(r-g-\gamma)\Delta}{1+\gamma\Delta},$$

which leads to

$$\frac{1}{c^\sigma}\frac{(c(s'))^\sigma - c^\sigma}{\Delta} = \frac{r-g-\gamma}{1+\gamma\Delta} = \frac{r-\rho-g\sigma}{1+\gamma\Delta}.$$

Thus, as we let $\Delta \to 0$,

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\sigma} - g$$

where I used

$$\frac{1}{c^\sigma}\frac{d(c^\sigma)}{dt} = \frac{d(\log(c^\sigma))}{dt} = \sigma\frac{d\log(c)}{dt} = \sigma\frac{\dot{c}}{c}.$$

Thus, in terms of non-detrended consumption C,

$$\frac{\dot{C}}{C} = \frac{r-\rho}{\sigma} \equiv \mu \tag{10}$$

$$\frac{\dot{u}(C)}{u(C)} = \frac{u'(C)}{u(C)}\dot{C} = (1-\sigma)\frac{\dot{C}}{C} = (1-\sigma)\mu \tag{11}$$

when the participation constraint doesn't bind. The assumption (2) implies that $r > \mu$.

When the participation constraint binds,

$$v'(s') = v^{Aut}(s') \tag{12}$$

must hold. If this is caused by the switch of state from low to high, there will be a discrete increase in consumption. Conversely, when the participation constraint in the same state continues to bind, the paths of C and $u(C)$ are described as

$$\frac{\dot{C}}{C} = g \tag{13}$$

$$\frac{\dot{u}(C)}{u(C)} = (1-\sigma)g \tag{14}$$

because endowment and the value of autarky respectively grows at rate g and $(1-\sigma)g$ when the state remains unchanged.

Figure 1 depicts the path of consumption under $g > \mu$, which turns out to be the more important case in this paper. In the figure, the fine solid curves correspond to $C_H(t)$ and $C_L(t)$, where $C_s(t)$ denotes consumption, at date t and in state s, that would be assigned by an optimal contract which provides $V^{Aut}(t, s(t))$ to the household at such date and state. Thus, when the path of optimal consumption intersects with the curve of $C_s(t)$, the contract provides the household exactly $V^{Aut}(t, s(t))$.

The path of consumption C is denoted by dashed lines when the participation constraint is slack, and by thick solid lines when it is binding. As we observe from the figure, the consumption path is divided into two phases: Phase 1, during which the participation constraint is slack regardless of the current state, and Phase 2, during which at least the participation constraint in the high state binds. Which of the two phases the consumption path starts from depends on the initial promised value to the household, V. When V is relatively large, consumption starts in Phase 1, as in Figure 1.

Figure 2 shows the path of consumption under $g \leq \mu$, which is again divided into two phases: Phase 1', during which the participation constraint may bind, and Phase 2', during which it is slack regardless of the current state. The only situation in which the participation constraint may bind in Phase 1' is either at the beginning of the contract, or when the state switches from low to high. In either case, unlike under $g > \mu$, the participation may bind only instantaneously. For, since the growth rate of optimal consumption (μ) exceeds that of endowment (g), after the participation constraint binds, the growth rate of consumption provided in the contract will exceed that of endowment, or the consumption under autarky, leaving no reason for the household to walk away from the contract.

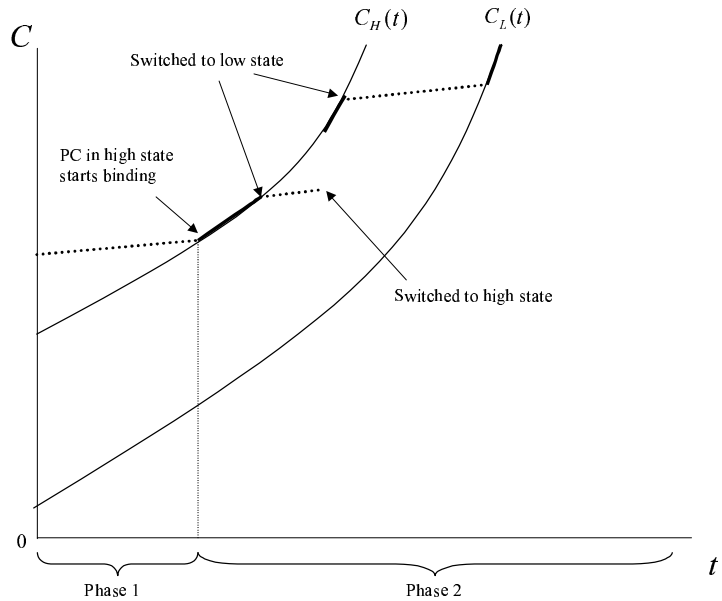

Figure 1: Consumption Path ($g > \mu$)

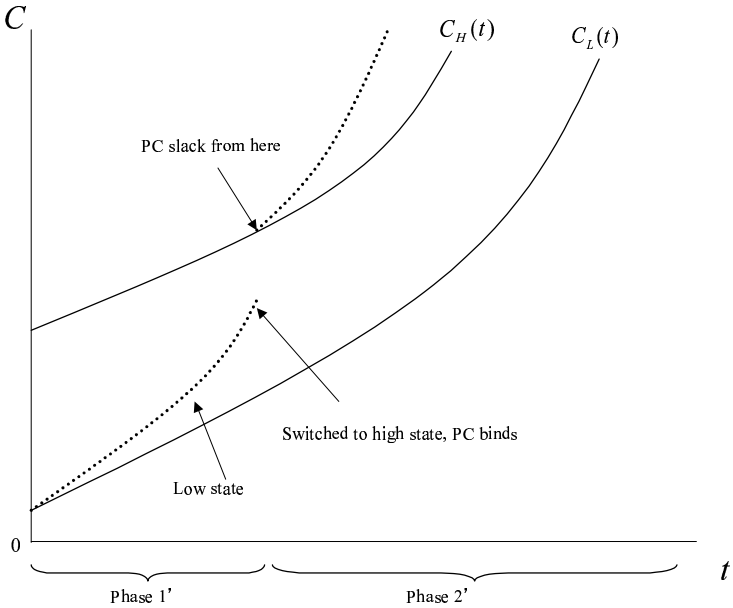

Figure 2: Consumption Path ($g \leq \mu$)

3 Uncertainty and Welfare - Analytical Approach

3.1 Effect of Growth Rate

I now move on to our central question of whether uncertainty can improve welfare. Let $V^U(\beta)$ denote the household's ex-ante welfare, or the expected present value of lifetime utility before the realization of the state at $t = 0$, provided by an optimal zero profit contract. Also, let V^C denote the corresponding welfare when there is no uncertainty, that is, $V^C \equiv V^U(\frac{1}{2})$. The formal definition of welfare-improving uncertainty is as follows.

Definition 1 *Uncertainty is welfare improving if $V^U(\beta) > V^C$ for some $\beta \in (\frac{1}{2}, 1]$.*

The following proposition gives a necessary condition for uncertainty to be welfare improving.

Proposition 2 *Uncertainty can be welfare improving only when $g > \mu$.*

Proof. Suppose $g \leq \mu$, and first assume that the household can commit to the contract. It is straightforward to derive that the optimal path of consumption is $\{C(t)\}$, $t = [0, \infty)$, that satisfies

$$\frac{\dot{C}(t)}{C(t)} = \mu$$

$$\int_0^\infty e^{-(r-\mu)t} C(0) dt = E_0 \int_0^\infty e^{-(r-g)t} Y(t, s(t)) dt$$

, which is independent of the value of β or the realization of state. Now, consider what happens under the certainty case $\beta = \frac{1}{2}$, when the household cannot commit to the contract. Since the optimal contract backloads the household's consumption relative to its endowment under $g \leq \mu$, it is never beneficial for the household to walk away from the contract. Therefore, the first best consumption profile can be achieved under certainty. On the other hand, under uncertainty, or equivalently $\beta > \frac{1}{2}$, such a consumption profile may violate the participation constraint at some t, and even if it never does, there can be no welfare improvement over the certainty case, which achieves the first best[4]. ■

Proposition 2 implies that $g > \mu$ is a necessary condition for uncertainty to be welfare improving. Thus, in most of the following analysis, I consider the case in which

$$g > \mu \tag{15}$$

[4]Typical model of limited commitment adopt discrete time and assume $\beta(1 + r) = 1$ and $g = 0$, where β denotes the household's discount factor. In discrete time, μ turns out to equal $[\beta(1 + r)]^{1/\sigma} - 1$, so under the assumptions above, $g = \mu = 0$ holds and hence the first best is achieved under no uncertainty.

holds. In this case, the optimal zero profit contract under certainty has a very different form.

Proposition 3 *When $g > \mu$ and $\beta = \frac{1}{2}$, the financial intermediary simply returns the endowment stream to the household under the optimal zero profit contract, so $V^U(\beta)$ equals the household's autarky value under certainty.*

Proof. Let V_C^{Aut} denote the household's autarky value under certainty at $t = 0$, and consider an optimal contract that provides this value to the household. Since $g > \mu$, if the participation constraint is slack, consumption grows slower than endowment. But given that the household's value equals V_C^{Aut} at $t = 0$ by assumption, under such a consumption profile, the household's continuation value from the contract will be lower than the value of autarky for $t > 0$. Therefore, the participation constraint will always bind for $t > 0$, which in turn implies that consumption will equal endowment $(\frac{1}{2})e^{gt}$ for all t. Since such a contract obviously provides zero profit to the financial intermediary, the proposition follows. ■

Proposition 2 implies that

$$V_C = V_C^{Aut} = \int_0^\infty e^{-\rho t} e^{g(1-\sigma)t} u(\frac{1}{2}) dt = \frac{1}{\gamma} u(\frac{1}{2}) \tag{16}$$

When $g > \mu$, an optimal contract tries to frontload consumption relative to endowment. So typically, consumption is initially relatively high compared to endowment, and will become lower later on. But under lack of commitment, the household will walk away from the contract after the initial high-consumption phase, and in order to prevent this, the contract must provide the household at least its autarky value during the low-consumption phase. But when there is no uncertainty, this makes the financial intermediary unable to make profits in the low-consumption phase, and as a consequence, the high-consumption phase cannot exist under the zero profit condition.

This is different when there is uncertainty, because the risk-sharing role of contract makes the contract valuable to the household even in the low-consumption phase, which allows the financial intermediary to make profits in the late stages of the contract. As a result, some frontloading of consumption can be achieved, which may potentially improve welfare compared to the certainty case. The next question is under what additional conditions uncertainty is indeed welfare improving.

3.2 Analytical Approach

Local Welfare Improvement

I first approach this question analytically, which yields a simple sufficient condition for uncertainty to be welfare improving. I start by discussing a condition under which uncertainty locally improves welfare.

Proposition 4 *If $g > \mu$, there exists $\beta_0 \in [\frac{1}{2}, 1)$ such that $V^U(\beta)$ is increasing in β for $\beta \in (\beta_0, 1)$.*

Proof. From (7), we obtain

$$\frac{\partial v^{Aut}(H)}{\partial \beta} = \frac{(\lambda+\gamma)u'(\beta) - \lambda u'(1-\beta)}{\gamma(2\lambda+\gamma)} \tag{17}$$

$$\frac{\partial v^{Aut}(L)}{\partial \beta} = \frac{-(\lambda+\gamma)u'(1-\beta) + \lambda u'(\beta)}{\gamma(2\lambda+\gamma)} \tag{18}$$

Notice that $\frac{\partial v^{Aut}(L)}{\partial \beta} < 0$ for any $\beta \in [\frac{1}{2}, 1]$, as $u'(1-\beta) \geq u'(\beta) > 0$ and $\gamma > 0$. On the other hand, $\frac{\partial v^{Aut}(H)}{\partial \beta} > 0$ at $\beta = \frac{1}{2}$, but $\frac{\partial v^{Aut}(H)}{\partial \beta} < 0$ for β sufficiently close to 1, because $\frac{\partial v^{Aut}(H)}{\partial \beta} < 0$ at $\beta = 1$ as $u'(0) = \infty$. As $\frac{\partial^2 v^{Aut}(H)}{\partial \beta^2} < 0$, $\frac{\partial v^{Aut}(H)}{\partial \beta} < 0$ if and only if $\beta > \beta_1$, where $(\lambda+\gamma)u'(\beta_1) = \lambda u'(1-\beta_1)$. This condition is equivalent to

$$\frac{\beta_1}{1-\beta_1} = (\frac{\lambda+\gamma}{\lambda})^{\frac{1}{\sigma}},$$

which can be rewritten as

$$\beta_1 = \frac{(\frac{\lambda+\gamma}{\lambda})^{\frac{1}{\sigma}}}{1 + (\frac{\lambda+\gamma}{\lambda})^{\frac{1}{\sigma}}}. \tag{19}$$

Since $\gamma > 0$, $\frac{\lambda+\gamma}{\lambda} > 1$ so that $\beta_1 \in (\frac{1}{2}, 1)$. Now, when $g > \mu$ and $\beta < 1$, endowment grows faster than consumption when the participation constraints do not bind, so these constraints can not be slack for all future states, which implies that relaxing them strictly improves welfare. When $\beta > \beta_1$, $\frac{\partial v^{Aut}(L)}{\partial \beta} < 0$ and $\frac{\partial v^{Aut}(H)}{\partial \beta} < 0$, so that an increase in β makes the household's participation constraints in both states less binding, so $V^U(\beta)$ has to be increasing in β for at least $\beta \in (\beta_1, 1)$. Thus, there exists $\beta_0 \in [\frac{1}{2}, \beta_1]$ such that $V^U(\beta)$ is increasing in β for $\beta \in (\beta_0, 1)$. ■

The RHS of (19) is increasing in γ. As $\gamma = \rho - g(1-\sigma)$, this means that the RHS is increasing in g if $\sigma > 1$, and decreasing in g if $\sigma < 1$. The intuition is as follows. A rise in β has two effects on $v^{Aut}(H)$: it increases $v^{Aut}(H)$ by increasing the current consumption, but decreases $v^{Aut}(H)$ by making the household worse off in the contingency of low state in

the future. Now, γ can be interpreted as an effective discount rate for a household. Thus, a larger value of γ implies that the values obtained in the future are discounted more heavily, or in other words, makes today relatively more important. Now, when $\sigma < 1$, the household's utility values are positive, and their absolute values are increasing in consumption. Thus, a larger value of g increases the absolute values of utility in the future, and makes them more important, which lowers the effective discount rate γ. This reinforces the latter effect on $v^{Aut}(H)$, and results in a fall in β_1, or equivalently in an expansion of the region of β for which $\frac{\partial v^{Aut}(H)}{\partial \beta} < 0$. Conversely, if $\sigma > 1$, utility values are negative, and a larger value of g reduces the absolute values of utility in the future. So this time, a rise in g increases the effective discount rate, and leads to the opposite implication on β_1.

Sufficient Condition for Welfare Improvement

It is obvious from the discussion above that, in order to find out whether uncertainty is welfare improving, it suffices to compare $V^C = V^U(\frac{1}{2})$ and $V^U(1)$. The following proposition derives a sufficient condition from this comparison.

Proposition 5 *Uncertainty is welfare improving if* (i) $g > \mu$ *and* (ii) $\sigma \geq 1 - \frac{\log(\frac{2\lambda+\gamma}{\lambda+\gamma})}{\log 2}$.

Proof. As mentioned earlier, (i) is a necessary condition for uncertainty to be welfare improving. (ii) is obtained by comparing the participation constraints under $\beta = \frac{1}{2}$ and $\beta = 1$, in both the high and low state. First, observe from (7) that both $v^{Aut}(L)$ and $v^{Aut}(H)$ equal $\frac{1}{\gamma}u(\frac{1}{2}) = V_C^{Aut}$ when $\beta = \frac{1}{2}$. Now, since $\frac{\partial v^{Aut}(L)}{\partial \beta} < 0$ for $\beta \in [\frac{1}{2}, 1]$, $v^{Aut}(L)$ under $\beta = 1$ is always strictly smaller than V_C^{Aut}. So if $v^{Aut}(H)$ under $\beta = 1$ is no greater than V_C^{Aut}, the financial intermediary's problem has less stringent participation constraints under $\beta = 1$, so that $V^C < V^U(1)$. This condition can be written as

$$\frac{(\lambda+\gamma)u(1) + \lambda u(0)}{2\lambda+\gamma} \leq \frac{1}{\gamma}u(\frac{1}{2}) \tag{20}$$

Clearly (20) is always satisfied when $\sigma > 1$, as $u(0) = -\infty$ in such a case. When $0 < \sigma < 1$, (20) becomes

$$\sigma \geq 1 - \frac{\log(\frac{2\lambda+\gamma}{\lambda+\gamma})}{\log 2}. \tag{21}$$

Since $\lambda > 0$ and $\gamma > 0$, $\log(\frac{2\lambda+\gamma}{\lambda+\gamma}) > 0$ so that $1 - \frac{\log(\frac{2\lambda+\gamma}{\lambda+\gamma})}{\log 2} < 1$. Thus, (21) is satisfied when $\sigma > 1$, so this serves as the general expression for the sufficient condition. ■

It requires caution that the RHS of ((21)) is itself a function of σ, as γ is so. The following corollary illustrates a special case, in which the sufficient condition allows an easier interpretation.

Corollary 6 *If $r = \rho > g > 0$, there exists $\sigma^* = \sigma^*(\lambda, \rho, g) \in (0,1)$ such that uncertainty is welfare improving if and only if $\sigma \geq \sigma^*$ and $\sigma \neq 1$.*

Proof. *Note that when $r = \rho > g > 0$, (2) and (15) are satisfied for any $\sigma > 0$. Also, as before, (20) is satisfied for all $\sigma > 1$. Thus, it suffices to examine when (21) holds under $0 < \sigma < 1$. Define $f(\sigma) \equiv \sigma - 1 + \frac{\log(\frac{2\lambda+\gamma}{\lambda+\gamma})}{\log 2}$. Obviously $f(\sigma)$ is continuous, and noting that $\gamma = \rho - g(1-\sigma)$, $f(\sigma) < 0$ as $\sigma \to 0$ and $f(\sigma) > 0$ as $\sigma \to 1$, so by the intermediate value theorem, there exists at least one $\sigma^* \in (0,1)$ such that $f(\sigma^*) = 0$. Now, $f'(\sigma) = \frac{1}{\log 2}(\log 2 - \frac{\lambda g}{(\lambda+\gamma)(2\lambda+\gamma)})$ and hence $f''(\sigma) > 0$, so it has to be that σ^* is unique and that $f(\sigma) \geq 0$ for all $\sigma \in [\sigma^*, 1)$. Thus, (20) holds if and only if $\sigma \geq \sigma^*$ and $\sigma \neq 1$, which completes the proof.* ■

This corollary implies that uncertainty is welfare improving for a sufficiently large σ. Intuitively, a rise in σ worsens the household's outside option values under uncertainty, and also increases the benefit of consumption frontloading enabled by the presence of uncertainty, both of which increase the value of uncertainty.

When the sufficient conditions above are satisfied, there exists $\beta_2 \in [\frac{1}{2}, 1)$ such that $V^U(\beta) > V_C$ for all $\beta > \beta_2$; β_2 is simply the largest β such that $V^U(\beta) = V_C$. But it is worth emphasizing that (21) is not necessary for uncertainty to be welfare improving. For, even if (20) is not satisfied and hence $v^{Aut}(H)$ under $\beta = 1$ is greater than that under $\beta = \frac{1}{2}$, $v^{Aut}(L)$ is always smaller under the former. In other words, the relaxation of the participation constraint in the low state may more than offset the tightening of it in the high state, and may result in welfare improvement. Thus, sharper predictions on the relation between uncertainty and welfare requires a numerical analysis, which I turn to in the next section. But before doing so, I summarize how λ, the arrival rate of the endowment shock, affects the household's welfare.

Arrival Rate of Shocks and Welfare

Proposition 7 *(1) The RHS of (ii) of the sufficient condition is decreasing in λ, and goes to 1 as $\lambda \to 0$, and to 0 as $\lambda \to \infty$.*

(2) If $g > \mu$, as $\lambda \to \infty$, $V^U(\beta) > V_C$ for all $\beta \in (\frac{1}{2}, 1]$ and $\sigma > 0$ ($\sigma \neq 1$).

(3) $V^U(\beta)$ is independent of λ for $\beta = \frac{1}{2}$, and also for $\beta = 1$ if $\sigma > 1$.

Proof. I omit the proof of 7(1) which is straightforward. For the proof of 7(2), notice from (7) that as $\lambda \to \infty$, both $v^{Aut}(H)$ and $v^{Aut}(L)$ approach $\frac{u(\beta)+u(1-\beta)}{2\gamma} < \frac{1}{\gamma}u(\frac{1}{2}) = V_C^{Aut}$, where the inequality follows from the strict concavity of $u(\cdot)$. Finally, for the proof of 7(3), notice from (7) that for $\beta = \frac{1}{2}$, λ drops out from the expression of both $v^{Aut}(H)$ and $v^{Aut}(L)$, and that if $\sigma > 1$, both $v^{Aut}(H)$ and $v^{Aut}(L)$ equal $-\infty$ for $\beta = 1$. ■

Intuitively, for a small λ, the value of autarky in the high state is relatively large because there is low probability of switching to the low state. Thus, for this value to be smaller than the value of autarky under no uncertainty, risk aversion has to be sufficiently large. As $\lambda \to 0$, this requires $\sigma > 1$, which yields $u(0) = -\infty$ and hence outweighs the decrease in the switching probability. On the contrary, for a large λ, the value of autarky in the high state is relatively low since there is high probability of switching to the low state. In the extreme case of $\lambda \to \infty$, the difference between $v^{Aut}(H)$ and $v^{Aut}(L)$ approaches zero, and both of them become smaller than the autarky value under no uncertainty for all $\sigma > 0$, not just for $\beta = 1$ but also for all $\beta \in (\frac{1}{2}, 1]$. So when $g > \mu$, as $\lambda \to \infty$, the household's welfare always becomes larger under uncertainty.

4 Uncertainty and Welfare - Numerical Approach

4.1 Numerical Analysis

Derivation of Welfare

The numerical analysis requires the derivation of equations, from which we may explicitly compute the household's welfare under an optimal zero profit contract. I defer the formal derivation to the Appendix, and instead provide its intuition here.

Recall the two phases illustrated in Figure 1. I first examine the optimal contract in Phase 2, and compute the expected profit of the financial intermediary at the beginning of Phase 2. If this is non-positive, we can obtain the equations that define the household's welfare just from the analysis of Phase 2. If this is positive, we need to proceed to analyze the contract in Phase 1, and combine the analyses in both phases.

Effect of Uncertainty on Welfare

The procedure above yields a set of equations that implicitly define the household's welfare, which allow us to perform a numerical analysis. While the following exercises should be considered as numerical examples rather than simulations based on full calibration, I pick some of the parameters, such as the discount rate and the interest rate, so that the time length of 1 can be interpreted as 1 year.

Figure 3 plots $V^U(\beta)$ for $\beta \in [\frac{1}{2}, 1)$, $g = 0.02$, $\rho = 0.04$, $r = 0.04$, $\lambda = 0.03$, and $\sigma = 0.5$. Figure 4 does the same, but uses $\sigma = 2$ instead. For comparison, both figures also plot $V_C = V^U(\frac{1}{2})$ under the same parameters.

In both examples, we observe that $V^U(\beta) < V^C$ for β close to $\frac{1}{2}$, and $V^U(\beta) > V^C$ for

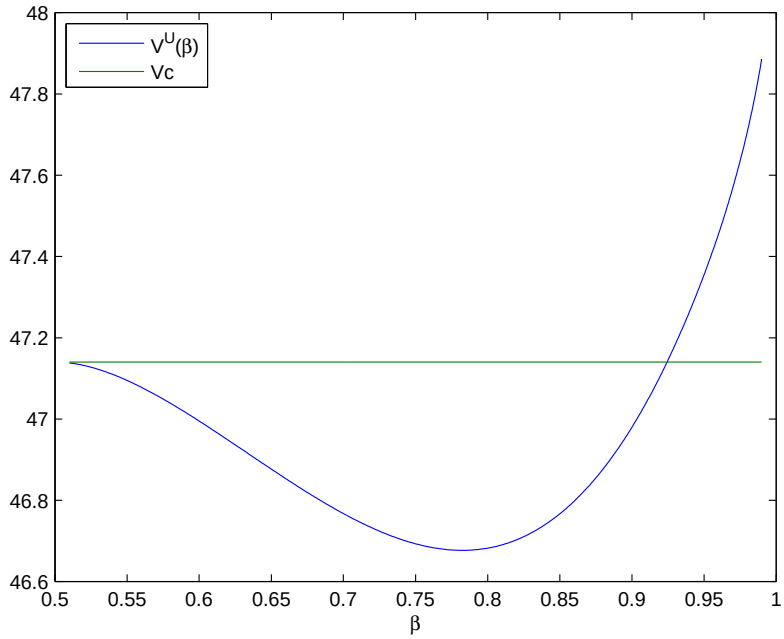

Figure 3: V^C and $V^U(\beta)$ ($\sigma = 0.5$)

sufficiently large β. Also, the range of β for which $V^U(\beta) > V^C$ is wider under $\sigma = 2$, which is natural given that the welfare gain from uncertainty stems from the household's consumption smoothing, whose value is larger under a larger σ.

How Large is the Welfare Gain/Loss ?

To facilitate the quantitative evaluation of the welfare effect of uncertainty, Figure 5 and 6 plot the consumption equivalent variations of the welfare gain or loss of $V^U(\beta)$, relative to V^C, for the same parameter values as in Figure 3 and 4. To be precise, the vertical axis corresponds to α expressed in percentage, where α solves

$$V^U(\beta) = \int_0^\infty e^{-\rho t} e^{g(1-\sigma)t} u(\frac{1+\alpha}{2}) dt = \frac{1}{\gamma} u(\frac{1+\alpha}{2}).$$

Figure 5 shows that when $\sigma = 0.5$, the welfare effect of uncertainty, either positive or negative, is not so large. Again this is natural, because when σ is small, the household's welfare cost of consumption fluctuation is small. As Figure 6 shows, however, the welfare gain from uncertainty may be fairly large when $\sigma = 2$, exceeding 30% of flow consumption in the certainty case, as β approaches 1.

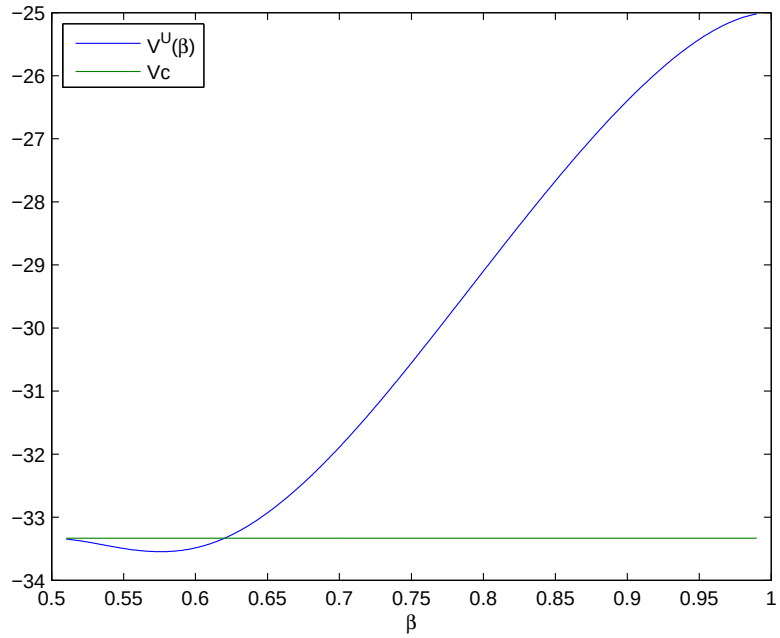

Figure 4: V^C and $V^U(\beta)$ $(\sigma = 2)$

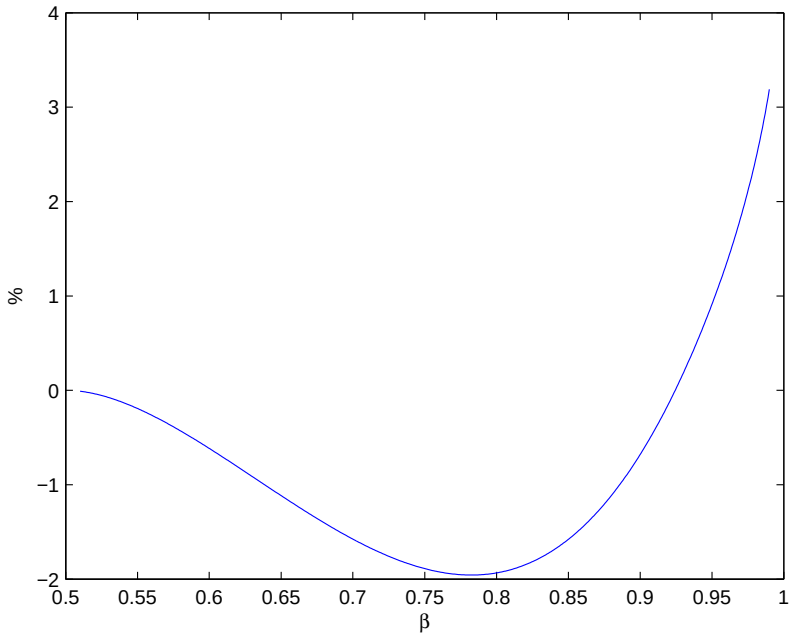

Figure 5: Welfare gain/loss relative to certainty case
(CE Variations in %, $\sigma = 0.5$)

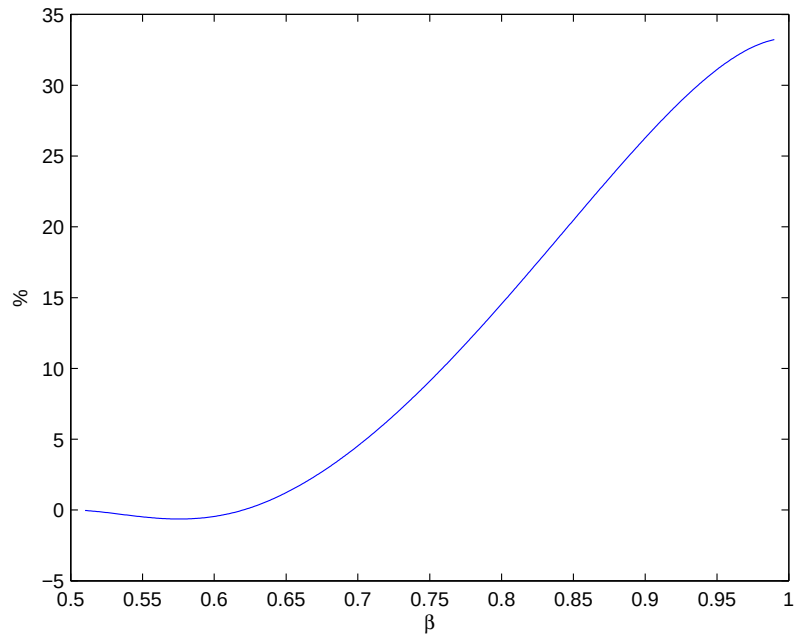

Figure 6: Welfare gain/loss relative to certainty case
(CE Variations in %, $\sigma = 2$)

Cost of Slower Growth

The results above suggest that the difference in growth rates may have a disproportionately large effect on welfare. The next two figures show this more explicitly.

Figure 7 uses $\rho = 0.04$, $r = 0.04$, $\lambda = 0.03$, and $\sigma = 2$ as in Figure 4 and 6, and plots $V^U(\beta)$ for $g = 0.01$ and $g = 0.03$. Again for comparison, V_C, for both parameter combinations, are plotted together. We observe that for large enough values of β, $V^U(\beta) > V_C$ for both values of g, but that the positive effect of uncertainty on welfare is much smaller under $g = 0.01$.

In Figure 8, to facilitate the evaluation of the magnitude of this effect, I again express the difference in welfare in terms of consumption equivalent variations. To be precise, I first compute the welfare gain or loss, relative to V_C for $g = 0.01$ and in terms of consumption equivalent variations, of the other series. I then take the difference of the series corresponding to $V^U(\beta)$ for $g = 0.03$ and $g = 0.01$, and plot it along with the series for V_C under $g = 0.03$. The resulting figure shows how the cost of 2% slower annual growth, converted into the fraction of consumption flows in the certainty case under $g = 0.01$, is affected by the size of uncertainty. Notice that the cost becomes very large as β becomes large, or as the degree

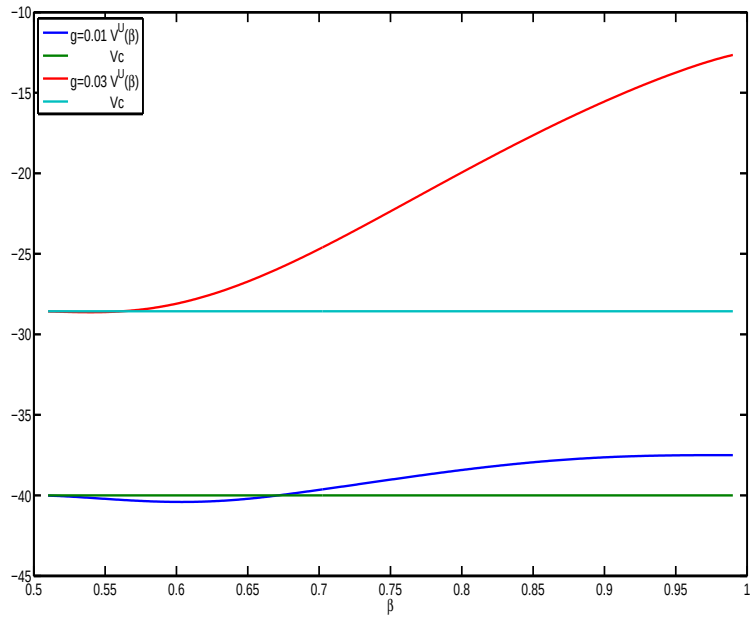

Figure 7: Growth, Uncertainty and Welfare

of uncertainty rises. In this example, the cost of 2% lower growth rate, for β close to 1, is about five times as large as that for $\beta = \frac{1}{2}$, or under no uncertainty.

A potential policy implication of these results is that, countries or entrepreneurs facing fast growth and volatile output may suffer very large welfare losses from growth slowdown. This is because, under the environment of the model, slow growth has an indirect cost of weakening or even eliminating the positive effect of uncertainty on welfare. In other words, while uncertainty may be a beneficial commitment device under fast growth, it becomes a nuisance when the growth rate becomes low, as in a more standard environment.

Arrival Rate of Shocks and Welfare

At the end of the previous section, I showed that the greater the arrival rate of the endowment shock λ, the easier it becomes to satisfy the sufficient condition for uncertainty to be welfare improving. Here, I analyze the quantitative impact of λ on the household's welfare.

Figure 9 and 10 plot the welfare gain or cost of uncertainty, again measured in consumption as in Figure 5 and 6, for three different values of λ, $\lambda = \{0.03, 0.1, 0.25\}$. The remaining parameters are the same as in Figure 5 and 6: $g = 0.02$, $\rho = 0.04$, $r = 0.04$, and $\sigma = 0.5$ in Figure 9, and $\sigma = 2$ in Figure 10. As Proposition 7(3) states, $V^U(\beta)$ is independent of

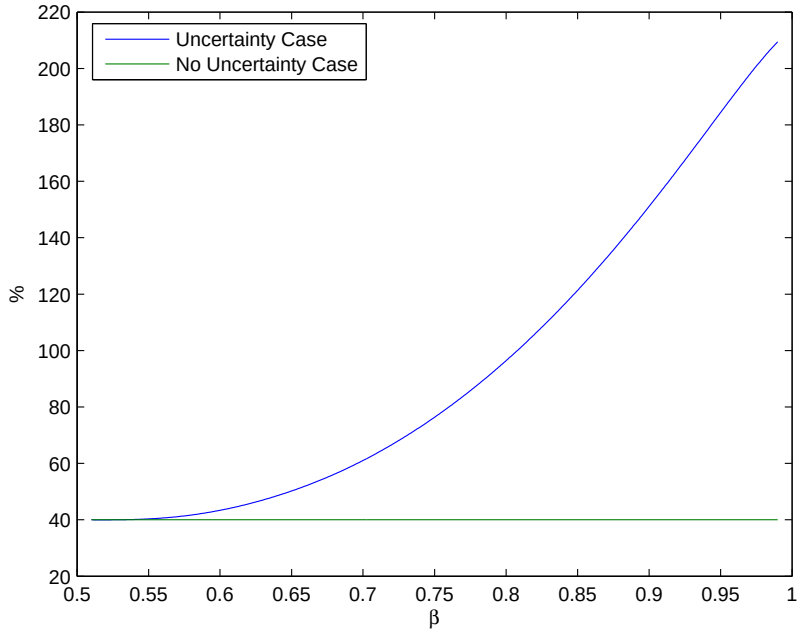

Figure 8: Welfare Cost of Slower Growth
(Uncertainty Case vs. No Uncertainty Case)

λ for $\beta = \frac{1}{2}$ for both values of σ, and also for $\beta = 1$ when $\sigma = 2 > 1$. For intermediate values of β, we observe in both figures that $V^U(\beta)$ is increasing in λ. While it is difficult to provide a general proof, such a positive welfare effect of λ appears fairly robust to the choice of parameters. From (7), it can be shown that $\frac{\partial v^{Aut}(H)}{\partial \lambda} < 0$ and $\frac{\partial v^{Aut}(L)}{\partial \lambda} > 0$, which implies that a larger λ relaxes the participation constraint in the high state, and tightens that in the low state. The positive welfare effect of λ indicates that the former effect in general dominates the latter.

5 Conclusion

In this paper, I examined the welfare implications of uncertainty in a simple financial contracting problem between a risk-neutral financial intermediary and a risk-averse household, when the latter cannot commit to the contract. Through both analytical and numerical analysis, I showed that if the growth rate of household's endowment is relatively high, the presence of uncertainty in the household's endowment may improve its welfare compared to the no-uncertainty case. This was because in such an environment, uncertainty makes the contract valuable by providing it the role of risk-sharing, which in turn helps achieving

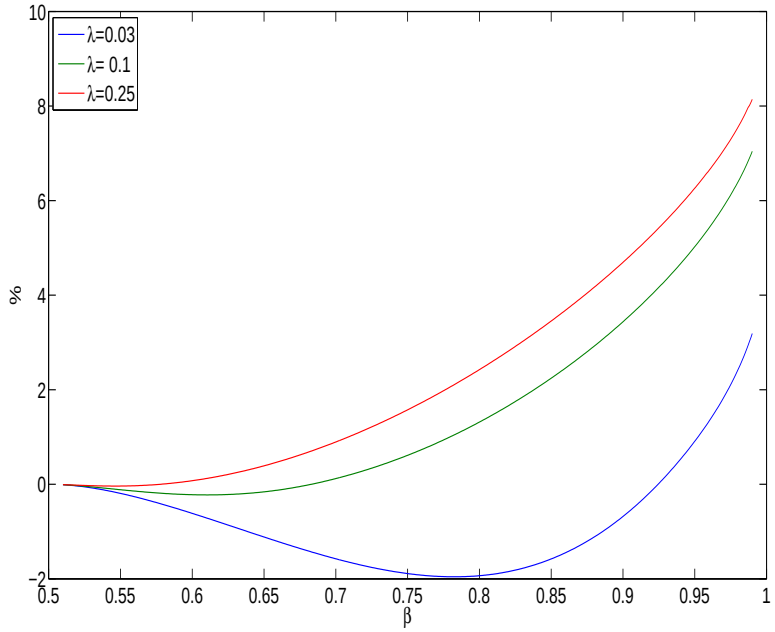

Figure 9: Welfare Effect of λ ($\sigma = 0.5$)

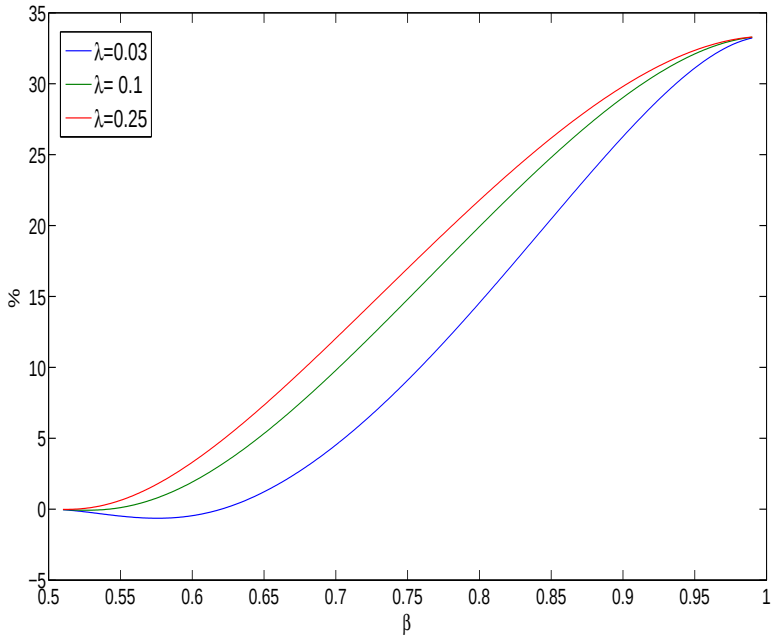

Figure 10: Welfare Effect of λ ($\sigma = 2$)

better intertemporal consumption smoothing by relaxing the commitment problem. I also showed that slower growth may have a disproportionately large effect on welfare as it dampens the positive effect of uncertainty on welfare. This may have interesting implications on, for example, how developing economies respond to the end of high-growth "catch-up" phase. Finally, I analyzed how the arrival rate of endowment shocks affects the welfare. I found that if the growth rate of household's endowment exceeds a certain level, uncertainty always improves welfare as the arrival rate approaches infinity. Moreover, I showed through a numerical analysis that the household's welfare is in general increasing in the arrival rate of shocks.

There are several potential extensions and applications of the paper. On the theoretical side, the current version of the model assumes CRRA utility for the household. While this is a convenient assumption that makes the analysis fairly tractable, it obscures the respective importance of the intertemporal elasticity of substitution and the degree of risk aversion in the welfare implication. I am currently thinking of extending the model by adopting other preferences that allow disentangling them, for example, the Epstein-Zin utility (Epstein and Zin (1989)). Also, while the current model assumes that the interest rate is exogenously given as in a small open economy model, I conjecture that the main qualitative results of the paper will hold in a closed-economy OLG model in which the interest rate is determined endogenously. On the empirical side, the model framework may be used to explore the implications of growth in topics to which limited commitment approach has been applied, such as international risk sharing under risk of default, and implicit insurance arrangements in village economies[5]. I leave such applications for future research.

[5]See e.g. Kehoe and Perri (2002) for the former application, and Ligon, Thomas, and Worrall (2002) for the latter.

Appendix: Computation of Welfare

Phase 2: Consumption and Welfare

Let us first focus on Phase 2, and define $c_L \equiv e^{-gt}C_L(t)$ and $c_H \equiv e^{-gt}C_H(t)$. Since consumption grows at rate g when the participation constraint continues to bind, and since the state switches from high to low with the Poisson arrival rate λ, $v^{Aut}(H)$ can be expressed as

$$v^{Aut}(H) = \int_0^\infty e^{-(\lambda+\rho)t} e^{g(1-\sigma)t}\{u(c_H) + \lambda v_L(0)\}dt$$

so that

$$u(c_H) = (\lambda+\gamma)v^{Aut}(H) - \lambda v_L(0) \tag{22}$$

Here, $v_L(\tau)$ denotes the detrended value provided to the household, who is in the low state and whose participation constraint is slack, where $\tau \geq 0$ denotes the time elapsed in Phase 2 since entering the low state most recently. That is, $\tau = 0$ either at the first date in Phase 2, or at later dates when the state switches from high to low. Define $T > 0$ so that the participation constraint in the low state starts to bind at $\tau = T$. By definition, $v_L(T) = v^{Aut}(L)$. Then, $v_L(\tau)$ can be expressed as

$$\begin{aligned}
v_L(\tau) &= \int_\tau^T e^{-(\lambda+\rho)(t-\tau)}\{e^{\mu(1-\sigma)t}e^{-g(1-\sigma)\tau}u(c_H) + \lambda e^{g(1-\sigma)(t-\tau)}v^{Aut}(H)\}dt + e^{-(\lambda+\rho-g(1-\sigma))(T-\tau)}v^{Aut}(L) \\
&= \int_\tau^T e^{-(\lambda+\gamma)(t-\tau)}\{e^{(\mu-g)(1-\sigma)t}u(c_H) + \lambda v^{Aut}(H)\}dt + e^{-(\lambda+\gamma)(T-\tau)}v_L^{Aut} \\
&= \frac{e^{(\mu-g)(1-\sigma)\tau}\{1-e^{-(\lambda+\eta)(T-\tau))}\}}{\lambda+\eta}u(c_H) + \frac{\lambda\{1-e^{-(\lambda+\gamma)(T-\tau)}\}}{\lambda+\gamma}v^{Aut}(H) + e^{-(\lambda+\gamma)(T-\tau)}v^{Aut}(L)
\end{aligned} \tag{23}$$

where $\eta \equiv \rho - \mu(1-\sigma)$.

Note that the household's detrended consumption remains at c_L (equivalently, the non-detrended consumption grows at rate g) when the state remains at low, and jumps up to provide the value equal to that in autarky when the state switches to high. Since this consumption pattern is exactly the same as that in autarky, it has to be that $c_L = 1-\beta$. Therefore,

$$e^{\mu T}c_H = e^{gT}c_L = e^{gT}(1-\beta),$$

or equivalently

$$e^{\mu(1-\sigma)T}u(c_H) = e^{g(1-\sigma)T}u(1-\beta), \tag{24}$$

so that

$$u(c_H) = e^{(g-\mu)(1-\sigma)T} u(1-\beta) = e^{(g-\mu)(1-\sigma)T}[(\lambda+\gamma)v^{Aut}(L) - \lambda v^{Aut}(H)] \tag{25}$$

Substituting $\tau = 0$ in (23),

$$v_L(0) = \frac{1-e^{-(\lambda+\eta)T}}{\lambda+\eta} u(c_H) + \frac{\lambda\{1-e^{-(\lambda+\gamma)T}\}}{\lambda+\gamma} v^{Aut}(H) + e^{-(\lambda+\gamma)T} v^{Aut}(L) \tag{26}$$

Using (22), (25) and rearranging,

$$\begin{aligned}&(\lambda+\gamma)[\frac{\eta-\gamma}{(\lambda+\gamma)(\lambda+\eta)} + (\frac{1}{\lambda} + \frac{1}{\lambda+\eta})e^{(\lambda+\eta)T}]v^{Aut}(L)\\ =&[\frac{\lambda(\eta-\gamma)}{(\lambda+\eta)(\lambda+\gamma)} + (\frac{\lambda+\gamma}{\lambda} - \frac{\lambda}{\lambda+\gamma})e^{(\lambda+\gamma)T} + \lambda(\frac{1}{\lambda} + \frac{1}{\lambda+\eta})e^{(\lambda+\eta)T}]v^{Aut}(H)\end{aligned} \tag{27}$$

Using (7), this can be also written as

$$\frac{1}{\lambda}e^{(\lambda+\gamma)T}u(\beta) - [\frac{\eta-\gamma}{(\lambda+\gamma)(\lambda+\eta)} + \frac{2\lambda+\eta}{\lambda(\lambda+\eta)}e^{(\lambda+\eta)T} - \frac{1}{\lambda+\gamma}e^{(\lambda+\gamma)T}]u(1-\beta) = 0 \tag{28}$$

This equation implicitly defines T.

Phase 2: Financial Intermediary's Profit

The next step is to compute the financial intermediary's expected profit at the first date in Phase 2. Let $(\pi_H^{Aut}, \pi_L^{Aut})$ denote the financial intermediary's detrended expected present value of profit in Phase 2 and in each state, when the participation constraint is binding. These can be computed from

$$\pi_H^{Aut} = \int_0^\infty e^{-(\lambda+r-g)t}\{\beta - c_H + \lambda\pi_L(0)\}dt \tag{29}$$

$$\pi_L^{Aut} = \int_0^\infty e^{-(\lambda+r-g)t}\{(1-\beta) - c_L + \lambda\pi_H^{Aut}\}dt \tag{30}$$

First, (29) becomes

$$(\lambda+r-g)\pi_H^{Aut} = \beta - c_H + \lambda\pi_L(0) \tag{31}$$

Further, since $c_L = 1-\beta$, (30) becomes

$$\pi_L^{Aut} = \frac{\lambda}{\lambda+r-g}\pi_H^{Aut} \tag{32}$$

Let $\pi_L(\tau)$ denote the detrended expected present value of profit in Phase 2, where $\tau \geq 0$ again denotes the time elapsed in Phase 2 since entering the low state most recently. Then,

$$\begin{aligned}\pi_L(\tau) &= \int_\tau^T e^{-(\lambda+r)(t-\tau)} e^{-g\tau}\{e^{gt}(1-\beta) - e^{\mu t}c_H + \lambda e^{gt}\pi_H^{Aut}\}dt + e^{-(\lambda+r-g)(T-\tau)}\pi_L^{Aut} \\ &= \frac{1-e^{-(\lambda+r-g)(T-\tau)}}{\lambda+r-g}(1-\beta) + \frac{\lambda(1-e^{-(\lambda+r-g)(T-\tau)})}{\lambda+r-g}\pi_H^{Aut} \\ &- \frac{e^{(\mu-g)\tau}(1-e^{-(\lambda+r-\mu)(T-\tau)})}{\lambda+r-\mu}c_H + e^{-(\lambda+r-g)(T-\tau)}\pi_L^{Aut}\end{aligned}$$

Thus, in particular,

$$\begin{aligned}\pi_L(0) &= \frac{1-e^{-(\lambda+r-g)T}}{\lambda+r-g}(1-\beta) + \frac{\lambda(1-e^{-(\lambda+r-g)T})}{\lambda+r-g}\pi_H^{Aut} - \frac{1-e^{-(\lambda+r-\mu)T}}{\lambda+r-\mu}c_H + e^{-(\lambda+r-g)T}\pi_L^{Aut} \\ &= \frac{1-e^{-(\lambda+r-g)T}}{\lambda+r-g}(1-\beta) + \frac{\lambda}{\lambda+r-g}\pi_H^{Aut} - \frac{1-e^{-(\lambda+r-\mu)T}}{\lambda+r-\mu}c_H \qquad (33)\end{aligned}$$

Combining (31) and (33) and rearranging,

$$\pi_H^{Aut} = \frac{\lambda(\lambda+r-g)}{(2\lambda+r-g)(r-g)}\{\frac{1}{\lambda}\beta + \frac{1-e^{-(\lambda+r-g)T}}{\lambda+r-g}(1-\beta) - (\frac{1}{\lambda} + \frac{1-e^{-(\lambda+r-\mu)T}}{\lambda+r-\mu})c_H\} \quad (34)$$

By symmetry, the probability of being in high or low state at the first date in Phase 2 is $\frac{1}{2}$ each. In equilibrium, the zero profit condition holds so that the financial intermediary's expected profit at the first date in Phase 2 is given by

$$\begin{aligned}\bar{\pi} &= 0 \qquad\qquad \text{if } \pi_H^{Aut} + \pi_L(0) \leq 0 \ \ \text{(Case 1)} \qquad (35) \\ &= \frac{1}{2}\{\pi_H^{Aut} + \pi_L(0)\} \text{ if } \pi_H^{Aut} + \pi_L(0) > 0 \ \ \text{(Case 2)}\end{aligned}$$

Using (31) and (34),

$$\begin{aligned}&\pi_H^{Aut} + \pi_L(0) \\ &= \pi_H^{Aut} + \frac{\lambda+r-g}{\lambda}\pi_H^{Aut} - \frac{\beta-c_H}{\lambda} \\ &= \frac{\lambda+r-g}{r-g}[\frac{1-e^{-(\lambda+r-g)T}}{\lambda+r-g}(1-\beta) + \frac{1}{\lambda}\beta - (\frac{1}{\lambda} + \frac{1-e^{-(\lambda+r-\mu)T}}{\lambda+r-\mu})e^{(g-\mu)T}(1-\beta)] - \frac{\beta - e^{(g-\mu)T}(1-\beta)}{\lambda} \\ &= \frac{1}{r-g} - \frac{[(\lambda+r-g)+(\lambda+r-\mu)]e^{(g-\mu)T} + (g-\mu)e^{-(\lambda+r-g)T}}{(\lambda+r-\mu)(r-g)}(1-\beta) \qquad (36)\end{aligned}$$

so that the condition $\pi_H^{Aut} + \pi_L(0) > 0$ is equivalent to

$$1 > \frac{[(\lambda+r-g)+(\lambda+r-\mu)]e^{(g-\mu)T} + (g-\mu)e^{-(\lambda+r-g)T}}{\lambda+r-\mu}(1-\beta) \qquad (37)$$

In Case 1, the household receives $v^{Aut}(H)$ at the initial date in Phase 2 if the state is high, and $v_L(\tilde{\tau}) < v_L(0)$ if the state is low, where $\tilde{\tau}$ is such that

$$\pi_H^{Aut} + \pi_L(\tilde{\tau}) = 0 \tag{38}$$

In contrast, in Case 2, the household receives $v^{Aut}(H)$ at the beginning of Phase 2 if the state is high, and $v_L(0)$ if the state is low. This case is discussed in detail in the next section.

Optimal Zero Profit Contract

I now provide a full characterization of an optimal zero profit contract, respectively for Case 1 and 2, and derive the corresponding expressions for the household's welfare.

Case 1: $\pi_H^{Aut} + \pi_L(0) \leq 0$

In this case, an optimal zero profit contract starts in the middle of Phase 2. $V^U(\beta)$ is given by

$$V^U(\beta) = \frac{1}{2}(v^{Aut}(H) + v_L(\tilde{\tau})) \tag{39}$$

, where $\tilde{\tau}$ is defined by (38).

Case 2: $\pi_H^{Aut} + \pi_L(0) > 0$

When $\pi_H^{Aut} + \pi_L(0) > 0$, an optimal zero profit contract starts from Phase 1, during which the participation constraint for neither state binds. Thus, a full description of the contract requires the analysis of this phase.

Let c_0 denote the initial consumption in Phase 1. Note that during Phase 1, consumption grows at rate μ regardless of the current state. Suppose the participation constraint in the high state starts to bind after T_1. Then,

$$c_0 e^{\mu T_1} = c_H e^{gT_1} \tag{40}$$

or equivalently,

$$u(c_0)e^{\mu(1-\sigma)T_1} = u(c_H)e^{g(1-\sigma)T_1}$$

The other condition to consider is the zero profit condition for the financial intermediary. Let $w_H(\tau)$ and $w_L(\tau)$ denote the detrended expected present value of endowment in Phase 1 at $t = \tau$. Since

$$w_H(\tau) = \int_\tau^{T_1} e^{-(r-g)(t-\tau)}\{\beta + \lambda[w_L(t) - w_H(t)]\}dt$$

$$w_L(\tau) = \int_\tau^{T_1} e^{-(r-g)(t-\tau)}\{(1-\beta) + \lambda[w_H(t) - w_L(t)]\}dt$$

it follows that

$$\frac{w_H(\tau)+w_L(\tau)}{2}=\frac{1}{2}\int_{\tau}^{T_1}e^{-(r-g)(t-\tau)}\{\beta+(1-\beta)\}dt=\frac{1-e^{-(r-g)(T_1-\tau)}}{2(r-g)}$$

On the other hand, the present value of consumption stream in Phase 1 is

$$\int_0^{T_1}e^{-(r-\mu)t}c_0dt=\frac{1-e^{-(r-\mu)T_1}}{r-\mu}c_0=\frac{1-e^{-(r-\mu)T_1}}{r-\mu}c_He^{(g-\mu)T_1}$$

Therefore, the financial intermediary's expected profit, before knowing the state at $t=0$, is given by

$$\frac{w_H(0)+w_L(0)}{2}+e^{-(r-g)T_1}\bar{\pi}=\int_0^{T_1}e^{-(r-\mu)t}c_0dt,$$

which can be simplified as

$$\frac{e^{(r-g)T_1}-1}{2(r-g)}-\frac{e^{(r-\mu)T_1}-1}{r-\mu}c_H+\frac{1}{2}(\pi_H^{Aut}+\pi_L(0))=0 \tag{41}$$

From (36), this condition becomes

$$e^{(r-g)T_1}+e^{(g-\mu)T}[\frac{2(r-g)(1-e^{(r-\mu)T_1})}{r-\mu}-\frac{(\lambda+r-g)+(\lambda+r-\mu)+(g-\mu)e^{-(\lambda+r-\mu)T}}{\lambda+r-\mu}](1-\beta)=0 \tag{42}$$

(42) implicitly defines T_1. Using T_1, the household's expected lifetime utility can be expressed as,

$$\begin{aligned}V^U(\beta)&=\int_0^{T_1}e^{-\eta t}u(c_0)dt+\frac{1}{2}e^{-\gamma T_1}(v^{Aut}(H)+v_L(0))\\&=\frac{1-e^{-\eta T_1}}{\eta}u(c_H)e^{(g-\mu)(1-\sigma)T_1}+\frac{1}{2}e^{-\gamma T_1}(v^{Aut}(H)+v_L(0))\end{aligned}$$

Using (7), (22) and (24),

$$\begin{aligned}v^{Aut}(H)+v_L(0)&=\frac{2\lambda+\gamma}{\lambda}\frac{(\lambda+\gamma)u(\beta)+\lambda u(1-\beta)}{\gamma(2\lambda+\gamma)}-\frac{e^{(g-\mu)(1-\sigma)T}u(1-\beta)}{\lambda}\\&=\frac{(\lambda+\gamma)u(\beta)+[\lambda-\gamma e^{(g-\mu)(1-\sigma)T}]u(1-\beta)}{\lambda\gamma}\end{aligned}$$

so that

$$V^U(\beta)=e^{-\gamma T_1}\frac{(\lambda+\gamma)}{2\lambda\gamma}u(\beta)+[\frac{1-e^{-\eta T_1}}{\eta}e^{(g-\mu)(1-\sigma)(T_1+T)}+e^{-\gamma T_1}\frac{\lambda-\gamma e^{(g-\mu)(1-\sigma)T}}{2\lambda\gamma}]u(1-\beta) \tag{43}$$

General Expression for Welfare

From (39) and (43), we can obtain a general expression for the household's welfare,

$$\begin{aligned} V^U(\beta) &= \int_0^{T_1} e^{-\eta t} u(c_0) dt + \frac{1}{2} e^{-\gamma T_1} (v^{Aut}(H) + v_L(\tilde{\tau})) \\ &= \frac{1 - e^{-\eta T_1}}{\eta} u(c_H) e^{(g-\mu)(1-\sigma)T_1} + \frac{1}{2} e^{-\gamma T_1} (v^{Aut}(H) + v_L(\tilde{\tau})) \end{aligned} \tag{44}$$

Notice that (39) is obtained by setting $\tilde{\tau} = 0$, and (43) is obtained by setting $T_1 = 0$.

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