Lecture 10: GLS, WLS, and FGLS

Generalized Least Square (GLS)

So far, we have been dealing with heteroskedasticity under OLS framework. But if we knew the variance-covariance matrix of the error term, then we can make a heteroskedastic model into a homoskedastic model.

As we defined before,
\[ E(uu') = \sigma^2 \Omega = \Sigma. \]

Define further that
\[ \Omega^{-1} = PP' \]

\( P \) is a “n x n” matrix

Pre-multiply \( P \) on a regression model

\[ Py = PX\beta + Pu \]

or
\[ \tilde{y} = \tilde{X}\beta + \tilde{u} \]

In this model, the variance of \( \tilde{u} \) is

\[ E(\tilde{u}\tilde{u}') = E(Puu'P') = PE(uu')P' = P\sigma^2\Omega P' = \sigma^2 PP\Omega P' = \sigma^2 I \]

Note that \( PP\Omega P' = I \), because define \( PP\Omega P' = A \), then \( P'P\Omega P' = P'A \). By the definition of \( P \), \( \Omega^{-1}\Omega P' = P'A \), thus \( P' = P'A \). Therefore, \( A \) must be \( I \).

Because \( E(\tilde{u}\tilde{u}') = \sigma^2 I \), the model satisfies the assumption of homoskedasticity. Thus, we can estimate the model by the conventional OLS estimation.

Hence,
\[
\hat{\beta} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y} \\
= (XX'P)^{-1}X'PPy \\
= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y
\]
is the efficient estimator of \( \beta \). This is called the **Generalized Least Square (GLS)** estimator. Note that the GLS estimators are unbiased when \( E(\hat{u} \mid X) = 0 \). The variance of GLS estimator is

\[
\text{var}(\hat{\beta}) = \sigma^2 (X' \hat{\Omega} X)^{-1} = \sigma^2 (X' \Omega^{-1} X)^{-1}.
\]

Note that, under homoskedasticity, i.e., \( \Omega^{-1} = I \), GLS becomes OLS.

The problem is, as usual, that we don’t know \( \sigma^2 \Omega \) or \( \Sigma \). Thus we have to either assume \( \Sigma \) or estimate \( \Sigma \) empirically. An example of the former is Weighted Least Squares Estimation and an example of the later is Feasible GLS (FGLS).

**Weighted Least Squares Estimation (WLS)**

Consider a general case of heteroskedasticity.

\[
\text{Var}(u_i) = \sigma_i^2 = \sigma^2 \omega_i.
\]

Then,

\[
E(u_iu_i') = \sigma^2 \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_n \end{bmatrix} = \sigma^2 \Omega, \quad \text{thus} \quad \Omega^{-1} = \begin{bmatrix} \omega_1^{-1} & 0 & 0 \\ 0 & \omega_2^{-1} & 0 \\ 0 & 0 & \omega_n^{-1} \end{bmatrix}.
\]

Because of \( \Omega^{-1} = P'P \), \( P \) is a n x n matrix whose \( i \)-th diagonal element is \( 1/\sqrt{\omega_i} \). By pre-multiplying \( P \) on \( y \) and \( X \), we get

\[
y^*_i = Py = \begin{bmatrix} y_1 / \sqrt{\omega_1} \\ y_2 / \sqrt{\omega_2} \\ \vdots \end{bmatrix} \quad \text{and} \quad X^*_i = PX = \begin{bmatrix} 1/\sqrt{\omega_1} x_{i1} / \sqrt{\omega_1} \ldots x_{ik} / \sqrt{\omega_1} \\ 1/\sqrt{\omega_2} x_{i1} / \sqrt{\omega_2} \ldots x_{2k} / \sqrt{\omega_2} \\ 1/\sqrt{\omega_n} x_{i1} / \sqrt{\omega_n} \ldots x_{nk} / \sqrt{\omega_n} \end{bmatrix}.
\]

The OLS on \( y^*_i \) and \( X^*_i \) is called the Weighted Least Squares (WLS) because each variable is weighted by \( \sqrt{\omega_i} \). The question is: where can we find \( \omega_i \)?
Feasible GLS (FGLS)

Instead of assuming the structure of heteroskedasticity, we may estimate the structure of heteroskedasticity from OLS. This method is called Feasible GLS (FGLS). First, we estimate $\hat{\Omega}$ from OLS, and, second, we use $\hat{\Omega}$ instead of $\Omega$.

$$\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}y$$

There are many ways to estimate FGLS. But one flexible approach (discussed in Wooldridge page 277) is to assume that

$$\text{var}(u \mid X) = u^2 = \sigma^2 \exp(\delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_k x_k)$$

By taking log of the both sides and using $u^2$ instead of $u^2$, we can estimate

$$\log(\hat{u}^2) = \alpha_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_k x_k + \epsilon.$$  

The predicted value from this model is $\hat{g}_i = \log(\hat{u}^2)$. We then convert it by taking the exponential into $\hat{\omega}_i = \exp(\hat{g}_i) = \exp(\log(\hat{u}^2)) = \hat{u}^2$. We now use WLS with weights $1/\hat{\omega}_i$ or $1/\hat{u}^2$.

Example 1

* Estimate the log-wage model by using WAGE1.dta with WLS
  * Weight is educ

  * Generate weighted varaibles
    . gen w=1/(educ)^0.5
    . gen wlogwage=logwage*w
    . gen wfemale=female*w
    . gen weduc=educ*w
    . gen wexper=exper*w
    . gen wexpsq=expsq*w

  * Estimate weighted least squares (WLS) model
    . reg wlogwage weduc wfemale wexper wexpsq w, noc

```
Source |       SS       df       MS                  Number of obs =     524
---------+------------------------------               F(  5,   519) = 1660.16
Model | 113.916451     5  22.7832901               Prob > F      =  0.0000
Residual | 7.12253755   519  .013723579               R-squared     =  0.9412
---------+------------------------------               Adj R-squared =  0.9406
Total | 121.038988   524  .230990435               Root MSE      =  .11715

wlogwage |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
```
Example 2

* Estimate reg
reg logwage educ female exper expsq
(Output omitted)
predict e, residual
gen logesq=ln(e*e)

reg logesq educ female exper expsq
(output omitted)
predict esqhat
(option xb assumed; fitted values)
gen omega=exp(esqhat)

* Generate weighted variables
.gen w=1/(omega)^0.5
.gen wlogwage=logwage*w
.gen wfemale=female*w
.gen weduc=educ*w
.gen wexper=exper*w
.gen wexpsq=expsq*w

* Estimate Feasible GLS (FGLS) model
.reg wlogwage weduc wfemale wexper wexpsq w, noc

End of Example 1

End of Example 2