Problem Set 1
Due in class on April 22

1. Question 1 (20 points)
Suppose a preference relation \( \succeq \) on \( X \) is rational. Then, show the followings.

(a) Reflexive: For any \( x \in X \), \( x \sim x \).
(b) Transitive: For any \( x, y, z \in X \), if \( x \succ y \) and \( y \succ z \), then \( x \succ z \).

where \( \sim \) and \( \succ \) are defined as follows:
\[
\begin{align*}
a \sim b & \iff a \preceq b \text{ and } b \preceq a \\
a \succ b & \iff a \preceq b \text{ and not } b \preceq a
\end{align*}
\]

2. Question 2 (50 points)
There are two goods, whose quantities are denoted by \( X \) and \( Y \), each being a real number. An individual’s consumption set consists of all \( (X, Y) \) such that \( X \geq 0 \) and \( Y > 2 \). His utility function is:
\[
U(X, Y) = \ln(X + 3) + \ln(Y - 2).
\]
The price of \( X \) is \( p \) and that of \( Y \) is \( q \); total income is \( I \). The aim of the question is to find the consumer’s demand functions and examine their properties. You need not worry about second-order conditions. Proceed as follows:

(a) First solve the problem by Lagrange’s method, ignoring the constraints \( X \geq 0 \), \( Y > 2 \). Show that the solutions for \( X \) and \( Y \) that you obtain are valid demand functions if and only if \( I \geq 3p + 2q \).

(b) Next suppose \( I \leq 3p + 2q \). Solve the utility maximization problem subject to the budget constraint and an additional constraint \( X \geq 0 \), using Kuhn-Tucker theory. Show that the solutions for \( X \) and \( Y \) you get here are valid demand functions if and only if \( 2q < I \leq 3p + 2q \). What happens if \( I \leq 2q \)?

In each of the following parts, consider the above cases (a) and (b) separately.

(c) Show that the demands are homogeneous of degree 0 in \( (p, q, I) \) jointly.
(d) Find the algebraic expressions for the income elasticities of demand for \( X, Y \). Which, if either, of the goods is a luxury?

(e) Find the marginal propensities to spend on the two goods. Which, if either, of the goods is inferior?

(f) Find the algebraic expressions for the own price derivatives \( \partial X / \partial p, \partial Y / \partial q \). Which, if either, of the goods is a Giffen good?

3. Question 3 (30 points)
(In this question, you can use Lagrange’s method taking for granted that the second-order conditions are satisfied and boundary solutions do not arise.)

There are two goods \( X \) and \( Y \), with prices \( p \) and \( q \). A consumer’s utility function is

\[
U(X, Y) = X^{1/3}Y^{2/3}.
\]

(a) Find algebraic expressions for the quantities that solve the usual problem:

\[
\max U(X, Y) \text{ s.t. } pX + qY \leq I
\]

These are functions of \( (p, q, I) \), and are called Marshallian demand functions. Denote them by \( X^m \) and \( Y^m \). Find the algebraic expression for the resulting utility \( u \) also as a function of \( (p, q, I) \).

(b) Now consider the mirror-image problem: how much income is needed to achieve at least a specified target utility level \( u \) if the consumer makes the most economical choices:

\[
\min pX + qY \text{ s.t. } U(X, Y) \geq u
\]

These are functions of \( (p, q, u) \), and are called Hicksian demand functions. Denote them by \( X^h \) and \( Y^h \). Find algebraic expressions for \( X^h \) and \( Y^h \).

(c) Evaluate \( \partial X^h / \partial q \) and \( Y^m \partial X^m / \partial I \). Show that the two are equal when \( u \) and \( I \) are related by the expression you found in (a) above.