1. Question 1 (20 points)
Suppose a preference relation $\succeq$ on $X$ is rational. Then, show the followings.

(a) Reflexive: For any $x \in X$, $x \sim x$. (10 points)

Since $\succeq$ is complete, $x \succeq x$ must hold. It is immediate that $x \succeq x$ (and $x \not\succeq x$) implies $x \sim x$.

(b) Transitive: For any $x, y, z \in X$, if $x \succ y$ and $y \succ z$, then $x \succ z$. (10 points)

By definition of strict preference $\succ$,

- $x \succ y \iff x \succeq y$ and not $y \succeq x$
- $y \succ z \iff y \succeq z$ and not $z \succeq y$

Since $\succeq$ is transitive, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$. Now, suppose $z \succeq x$. By $x \succeq y$ and transitivity, we must have $z \succeq y$, which contradicts to the assumption $(y \succ z)$. Therefore, $z \succeq x$ cannot happen. Note that $x \succeq z$ and “not $z \succeq x$” implies $x \succ z$, concluding the proof.

2. Question 2 (50 points)
There are two goods, whose quantities are denoted by $X$ and $Y$, each being a real number. An individual’s consumption set consists of all $(X, Y)$ such that $X \geq 0$ and $Y > 2$. His utility function is:

$$U(X, Y) = \ln(X + 3) + \ln(Y - 2).$$

The price of $X$ is $p$ and that of $Y$ is $q$; total income is $I$. The aim of the question is to find the consumer’s demand functions and examine their properties. You need not worry about second-order conditions. Proceed as follows:

(a) First solve the problem by Lagrange’s method, ignoring the constraints $X \geq 0$, $Y > 2$. Show that the solutions for $X$ and $Y$ that you obtain are valid demand functions if and only if $I \geq 3p + 2q$. (20 points)

Need to solve: $\max_{X,Y} U(X, Y)$, s.t. $pX + qY \leq I$. 


Let \( L := \ln(X + 3) + \ln(Y - 2) + \lambda(I - pX - qY) \) and solve the unconstrained maximization problem with respect to \( X, Y \) and \( \lambda \). FOCs are as follows:

\[
\frac{\partial L}{\partial X} = \frac{1}{X + 3} - \lambda p = 0 \quad (1)
\]
\[
\frac{\partial L}{\partial Y} = \frac{1}{Y - 2} - \lambda q = 0 \quad (2)
\]
\[
\frac{\partial L}{\partial \lambda} = I - pX - qY = 0 \quad (3)
\]

By (1) and (2), we can derive \( q(Y - 2) = p(X + 3) \). Substituting for \( Y \) into (3), we obtain

\[
I - pX - (pX + 3p + 2q) = 0
\]
\[
2pX = I - 3p - 2q
\]
\[
X = \frac{I - 3p - 2q}{2p}. \quad (4)
\]

Solving for \( Y \) by plugging equation (4) into (3) gives:

\[
Y = \frac{I + 3p + 2q}{2p}
\]

Thus, the demand as a function of the prices, \( p \) and \( q \), and income, \( I \), is:

\[
(X, Y) = (\frac{I - 3p - 2q}{2p}, \frac{I + 3p + 2q}{2p})
\]

which only makes sense if \( I \geq 3p + 2q \), so that the demand for each good is always non-negative. Note that the condition that \( Y > 2 \) is also satisfied whenever \( I \geq 3p + 2q \).

(b) Next suppose \( I \leq 3p + 2q \). Solve the utility maximization problem subject to the budget constraint and an additional constraint \( X \geq 0 \), using Kuhn-Tucker theory. Show that the solutions for \( X \) and \( Y \) you get here are valid demand functions if and only if \( 2q < I \leq 3p + 2q \). What happens if \( I \leq 2q \)? (10 points)

If \( I < 3p + 2q \), we see from (a) that unconstrained demand for \( X \) would be negative. If we impose that \( X \geq 0 \), we know from Kuhn-Tucker theory that the consumer will optimize demand by setting \( X = 0 \) exactly. Thus, he will reach his best utility outcome by purchasing good \( Y \) only. With income \( I \), he can purchase
I/q units of good Y. So his total demand is:

\[(X, Y) = (0, \frac{I}{q})\]

Note that this demand function holds only for \(2q < I \leq 3p + 2q\). If \(I > 3p + 2q\), we have the demand function as in (a).

If \(I \leq 2q\), then the consumer will not have enough money to buy even 2 units of Y, thus not allowing him to achieve any utility above negative infinity. In this case, we say that utility is not clearly defined (since \(\ln(Y-2)\) does not exist), so demand is not defined either.

(c) Show that the demands are homogeneous of degree 0 in \((p, q, I)\) jointly. (5 points)

Skip.

(d) Find the algebraic expressions for the income elasticities of demand for \(X, Y\). Which, if either, of the goods is a luxury? (5 points)

Note that an income elasticity for each good is calculated by:

\[\left(\frac{\partial X}{\partial I}, \frac{\partial Y}{\partial I}\right)\]

When \(I \geq 3p + 2q\), the income elasticity of demand is:

\[\left(\frac{I}{I-3p-2q}, \frac{I}{I+3p+2q}\right)\]

Since the former exceeds 1, good \(X\) is a luxury good. If \(2q < I \leq 3p + 2q\), the income elasticity of demand is:

\[(0, 1)\]

So, neither one is a luxury good in this case.

(e) Find the marginal propensities to spend on the two goods. Which, if either, of the goods is inferior? (5 points)

The marginal propensity to spend is just the first derivative of demand relative to income:

\[\left(\frac{\partial X}{\partial I}, \frac{\partial Y}{\partial I}\right) = \begin{cases} \left(\frac{1}{2p}, \frac{1}{2q}\right) & \text{if } I \geq 3p + 2q \\ \left(0, \frac{1}{q}\right) & \text{if } 2q < I \leq 3p + 2q \end{cases}\]

Note here that the first derivative w.r.t. income is everywhere non-negative. Thus, neither good is an inferior good.

(f) Find the algebraic expressions for the own price derivatives \(\partial X/\partial p, \partial Y/\partial q\). Which, if either, of the goods is a Giffen good? (5 points)
The price derivatives are as follows:

\[
\left( \frac{\partial X}{\partial p}, \frac{\partial Y}{\partial q} \right) = \begin{cases} 
\left( -\frac{I-2q}{2p^2}, -\frac{I+3p}{2q^2} \right) & \text{if } I \geq 3p + 2q \\
(0, -\frac{I}{q^2}) & \text{if } 2q < I \leq 3p + 2q
\end{cases}
\]

We see that the price derivatives are negative (and demand is equal to 0 at \( I = 3p + 2q \)), meaning that demand decreases in the price of each good. Thus, neither good is a Giffen good.

3. Question 3 (30 points)

(In this question, you can use Lagrange’s method taking for granted that the second-order conditions are satisfied and boundary solutions do not arise.)

There are two goods \( X \) and \( Y \), with prices \( p \) and \( q \). A consumer’s utility function is

\[
U(X,Y) = X^{1/3}Y^{2/3}.
\]

(a) Find algebraic expressions for the quantities that solve the usual problem:

\[
\max U(X,Y) \text{ s.t. } pX + qY \leq I
\]

These are functions of \((p,q,I)\), and are called Marshallian demand functions. Denote them by \( X^m \) and \( Y^m \). Find the algebraic expression for the resulting utility \( u \) also as a function of \((p,q,I)\). (15 points)

Using Lagrange’s method, you can obtain the following Marshallian demand functions:

\[
(X^m, Y^m) = \left( \frac{I}{3p}, \frac{2I}{3q} \right)
\]

Plugging these demands into the utility function, we get:

\[
u = U(X^m, Y^m) = (X^m)^{1/3}(Y^m)^{2/3}
\]

\[
= \left( \frac{I}{3p} \right)^{1/3} \left( \frac{2I}{3q} \right)^{2/3} = \frac{2^{2/3}}{3p^{1/3}q^{2/3}} I
\]

(b) Now consider the mirror-image problem: how much income is needed to achieve at least a specified target utility level \( u \) if the consumer makes the most economical choices:

\[
\min pX + qY \text{ s.t. } U(X,Y) \geq u
\]

These are functions of \((p,q,u)\), and are called Hicksian demand functions. Denote them by \( X^h \) and \( Y^h \). Find algebraic expressions for \( X^h \) and \( Y^h \). (10 points)
Again, Lagrange’s method leads you to obtain the following Hicksian demand functions:

\[(X^h, Y^h) = (u\left(\frac{q}{2p}\right)^{2/3}, u\left(\frac{2p}{q}\right)^{1/3})\]

(c) Evaluate \(\partial X^h/\partial q\) and \(Y^m \partial X^m/\partial I\). Show that the two are equal when \(u\) and \(I\) are related by the expression you found in (a) above. \(5\) points

By calculation,

\[
\frac{\partial X^h}{\partial q} = \frac{2^{4/3}}{3p^{2/3}q^{1/3}} u
\]

\[
Y^m \frac{\partial X^m}{\partial I} = \frac{2}{9pq} I
\]

Substituting \(u = \frac{2^{2/3}}{3p^{1/3}q^{2/3}} I\) in the first line, we obtain

\[
\frac{\partial X^h}{\partial q} = \frac{2}{9pq} I = Y^m \frac{\partial X^m}{\partial I}.
\]