Problem Set 3: Solutions
Due in class on May 27

1. Question 1 (20 points)

Consider an exchange economy with \( n \) consumers, and suppose their preferences are (strictly) monotonic. Then, show that a general equilibrium allocation \( x \) is Pareto efficient.

**Note:** You should NOT rely on intuitive graphical analysis or MRS argument. Instead, try to give a formal proof of the first welfare theorem.

By monotonicity, budget constraints for all the consumers must be satisfied with equalities. Now, suppose \( x \) is not Pareto efficient. Then, there must exist an alternative allocation \( x' \) such that \( x'_j \geq x_j \) for \( j = 1, \ldots, n \) and \( x'_i > x_i \) for some \( i \). Pick such consumer \( i \). Her budget spent for new consumption bundle must be strictly greater than her income (otherwise contradict to \( x_i \) being an optimal choice), that is,

\[
p x'_i > p x_i = p \omega_i.
\]

For other consumers \( j \neq i \), weak version of the above inequality must hold,

\[
p x'_j \geq p \omega_j.
\]

Summing over across agents, we obtain

\[
p \sum_{j=1}^{n} x'_j > p \sum_{j=1}^{n} \omega_j.
\]

However \( \sum_{j=1}^{n} x'_j = \sum_{j=1}^{n} \omega_j \) must hold since \( x' \) is a feasible allocation, which is a contradiction.

2. Question 2 (40 points)

A firm can hire at a rental price \( r \) and labor at a wage \( w \). To produce anything at all requires one unit of capital, i.e. \( r \) is a fixed cost; this is sunk in the short run, but not sunk in the long run. If in a unit of time the firm employs \( L \) units of labor, and rents
$K$ units of capital (in addition to the one unit needed as a fixed cost), its output $Q$ is given by one of the following production functions:

\[ Q = K^{1/4}L^{1/8} \tag{1} \]

\[ Q = \min(K, L) \tag{2} \]

For each production function, answer the following questions.

(a) In the short run, the firm is committed to hire a fixed amount of capital $K(+1)$, and can vary its output $Q$ only by employing an appropriate amount of labor $L$. Find algebraic expressions for the firm’s short-run total, average, and marginal cost functions.

\[ \text{Firm’s short run total cost function, } STC_K(r, w, Q) \text{ is the answer of the following cost minimization problem in which the firm can only change } L. \]

\[ \min_L r(1 + K) + wL \]

\[ \text{s.t. } Q(K, L) = Q \]

(1) $L = \left(\frac{Q}{K^{1/4}}\right)^8 = \frac{Q^8}{K^2}, \ STC_K(r, w, Q) = r(1 + K) + w\frac{Q^8}{K^2}, \ SAC_K(r, w, Q) = \frac{r(1+K)}{Q} + w\frac{Q^7}{K^2}, \text{ and } SMC_K(r, w, Q) = 8w\frac{Q^7}{K^2}.$

(2) $L = \begin{cases} Q & \text{if } Q \leq K \\ \text{no answer} & \text{if } Q > K \end{cases}$. Since there is no solution if $Q > K$, we restrict our attention only to $Q \leq K$. Then, $STC_K(r, w, \overline{Q}) = r(1+K) + wQ$, $SAC_K(r, w, \overline{Q}) = \frac{r(1+K)}{Q} + w$, and $SMC_K(r, w, \overline{Q}) = w$.

(b) In the long run, the firm can vary both capital and labor. Find algebraic expressions for the firm’s long-run total, average, and marginal cost functions.

\[ \text{Firm’s long run total cost function, } LTC(r, w, \overline{Q}) \text{ is the answer of the following cost minimization problem in which the firm can change both } L \text{ and } K. \]

\[ \min_{K, L} r(1 + K) + wL \]

\[ \text{s.t. } Q(K, L) = \overline{Q} \]

(1) Solving by Lagrange’s method, we obtain

\[ K = Q^{8/3}\left(\frac{2w}{r}\right)^{1/3}, \ L = Q^{8/3}\left(\frac{r}{2w}\right)^{2/3}. \]
Thus,

\[ \text{LTC}(r, w, Q) = r(1 + Q^{8/3}(\frac{2w}{r})^{1/3}) + wQ^{8/3}(\frac{r}{2w})^{2/3} \]
\[ = r + \frac{3\sqrt[3]{2}}{2}r^{\frac{2}{3}}w^{\frac{1}{3}}Q^{\frac{2}{3}} \]

\[ \text{LAC}(r, w, Q) = \frac{r}{Q} + \frac{3\sqrt[3]{2}}{2}r^{\frac{2}{3}}w^{\frac{1}{3}}Q^{\frac{2}{3}} \]
\[ \text{LMC}(r, w, Q) = 4\sqrt[3]{2}r^{\frac{2}{3}}w^{\frac{1}{3}}Q^{\frac{2}{3}}. \]

(2) From the production function, the optimal inputs must be \( K = L = Q \). Therefore,

\[ \text{LTC}(r, w, Q) = r(1 + Q) + wQ \]
\[ \text{LAC}(r, w, Q) = r\left(1 + \frac{Q}{Q}\right) + w \]
\[ \text{LMC}(r, w, Q) = r + w. \]

(c) To link the short-run and the long-run cost curves, take the short-run average cost curve, and for given \( Q \), find the \( K \) (as a function of \( Q \)) that minimizes short-run average cost. Substitute this in the short-run average cost function, reducing it to a function of \( Q, r \) and \( w \). Verify that it is the same as the long-run average cost function.

\textit{Let} \( K^*(Q) \) \textit{be the amount of} \( K \) \textit{that minimizes} \( SAC \).

(1) By FOC,

\[ \frac{\partial SAC}{\partial K} = \frac{r}{Q} - 2w \frac{Q^7}{K^{6/3}} = 0 \]
\[ \Rightarrow K^*(Q) = \left(\frac{2w}{r}\right)^{\frac{1}{3}}Q^{\frac{8}{3}}. \]
Substituting it into $SAC_K$,

$$SAC_K = \frac{r(1 + K)}{Q} + w \frac{Q^7}{K^2}$$

$$= \frac{r(1 + (\frac{2w}{r})^\frac{3}{2} Q^\frac{5}{2})}{Q} + w \frac{Q^7}{(\frac{2w}{r})^\frac{3}{2} Q^\frac{10}{3}}$$

$$= \frac{r}{Q} + \frac{3\sqrt{2}}{2} r^2 w^2 Q^\frac{5}{2} = LAC.$$  

(2) Since $\frac{\partial SAC_K}{\partial K} = \frac{r}{Q}$ is always positive, $K^*(Q)$ must be the smallest $K$ that can produce $Q$. That is, $K^*(Q) = Q$. Therefore, we obtain

$$SAC_{K^*} = \frac{r(1 + K^*)}{Q} + w = \frac{r(1 + Q)}{Q} + w = LAC.$$  

3. Question 3 (40 points)

Ann has an endowment of 200 units of good $X$ and 5 units of good $Y$. Bob has an endowment of 100 units of good $X$ and 5 units of good $Y$. Answer the following questions of each of the following two cases where $U_A$ is Ann’s utility function and $U_B$ is Bob’s utility function:

Case 1: $U_A(X_A, Y_A) = X_A Y_A$ and $U_B(X_B, Y_B) = X_B Y_B$

Case 2: $U_A(X_A, Y_A) = X_A + 30 Y_A$ and $U_B(X_B, Y_B) = \text{min}(X_B, 30 Y_B)$

Answer the following questions algebraically but illustrate your answers in an Edgeworth box.

(a) Describe the set of efficient allocations in this economy.

(1) $Y = \frac{X}{30}$

(2) $Y = \frac{X}{30}$

(b) Describe the set of allocations which “Pareto-improve,” i.e., make both individuals better off, on the endowment allocation.

(1) $\{X, Y \mid XY > 10000 \text{ and } (300 - X)(10 - Y) > 5000\}$

(2) $\{X, Y \mid X + 30Y \geq 350 \text{ and } \text{min}(X, 30Y) \geq 200\}$

(c) Describe the “contract curve.”

Contract curve is the intersection of your answers in (a) and (b). That is,

(1) $\{X, Y \mid Y = \frac{X}{30}, XY > 10000 \text{ and } (300 - X)(10 - Y) > 5000\}$

(2) $\{X, Y \mid Y = \frac{X}{30}, X + 30Y \geq 350 \text{ and } \text{min}(X, 30Y) \geq 200\}$.  

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Find a competitive equilibrium price vector and allocation for this economy.

(1) Deriving Marshallian demand functions and plug them into market clearing condition, we obtain $P_X/P_Y = 1/30$. Therefore, the demand under equilibrium prices for each consumer is

\[
(X_A, Y_A) = (175, \frac{35}{6})
\]

\[
(X_B, Y_B) = (125, \frac{25}{6}).
\]

(2) The same as (1).