Problem Set 5: Solutions
Due in class on July 8

1. Question 1 (30 points)
A monopolist faces two kinds of consumers: students and non-students. The demand curve of each student is $q = 100 - 2p$. The demand of each non-student is given as $q = 100 - p$. There are $x$ students and $y$ non-students. There is a zero marginal cost of production.

(a) First suppose that the monopolist must set a single price to sell to all consumers. What price would the monopolist charge? How much would each student consume? And each non-student?

The market demand is $Q = \begin{cases} 
  y(100-p) & \text{if } 50 \leq p \leq 100 \\
  y(100-p) + x(100-2p) & \text{if } p < 50
\end{cases}$

If $p \geq 50$ we have

$$\frac{d\pi}{dp} = y(100 - 2p) = 0 \Rightarrow$$

$$p = 50 \Rightarrow$$

$$\pi = 2500y$$

If $p < 50$ we have

$$\frac{d\pi}{dp} = y(100 - 2p) + x(100 - 4p) = 0 \Rightarrow$$

$$p = \frac{50(x + y)}{y + 2x} \Rightarrow$$

$$\pi = y \left(100 - \frac{50(x+y)}{y + 2x}\right) \frac{50(x+y)}{y + 2x} + x \left(100 - 2\frac{50(x+y)}{y + 2x}\right) \frac{50(x+y)}{y + 2x}$$

$$= \frac{2500}{2x+y} (x+y)^2.$$
Since \( y < \frac{(x+y)^2}{2x+y} \), the profit in the second case is higher. The final answer is

\[
p^* = \frac{50(x+y)}{2x+y}, \quad q_s = \frac{x}{2x+y}, \quad q_{ns} = \frac{50(3x+y)}{2x+y}.
\]

(b) Now suppose that the monopolist can charge different prices to students and non-students. What price would the monopolist charge in each market? How much would each student consume? And each non-student?

For non-students we have already showed that \( p_{ns} = 50 \) and \( q_{ns} = 50 \). For students

\[
\pi = y(100 - 2p) \Rightarrow \\
\frac{d\pi}{dp} = y(100 - 4p) = 0 \Rightarrow \\
p_s = 25, \\
q_s = 50.
\]

(c) Compute the social welfare in (a) and (b). When does the single price regulation (under (a)) generate higher welfare than in (b)?

Let \( SW_a \) and \( SW_b \) be the social welfare under (a) and (b). Each of them can be calculated by subtracting two triangle area from the maximum possible surplus.

\[
SW_a = y \left( \frac{100^2}{2} + x \frac{50 \cdot 100}{2} - y \frac{(p^*)^2}{2} - x \frac{2(p^*)^2}{2} \right) \\
= \frac{1}{2} \{10000y + 5000x - (y + 2x)(p^*)^2\} \\
= \frac{1}{2} \{10000y + 5000x - (y + 2x)\left(\frac{50(x+y)}{y+2x}\right)^2\} \\
= \frac{1}{2} \{10000y + 5000x - \frac{2500(x+y)^2}{y+2x}\} \\

And,

\[
SW_b = y \left( \frac{100^2}{2} + x \frac{50 \cdot 100}{2} - y \frac{(p_{ns})^2}{2} - x \frac{2(p_s)^2}{2} \right) \\
= \frac{1}{2} \{10000y + 5000x - y50^2 - x25 \cdot 50\}. 
\]
So, the difference between two is

\[ SW_a - SW_b = 1250 \left\{ \left( y - \frac{x}{2} \right) - \frac{(x + y)^2}{y + 2x} \right\} \]

\[ = \frac{1250}{2(y + 2x)} \left\{ (y + 2x)(2y + x) - 2(x + y)^2 \right\} \]

\[ = \frac{1250xy}{2(y + 2x)} \geq 0. \]

Therefore, the social welfare under (a) is always higher than that under (b), i.e., single price regulation improves social welfare.

2. Question 2 (40 points)

To produce output of a homogenous good, each firm must pay a fixed cost of $f$ and a marginal cost of $c$ per unit. The demand curve for this good is \( p = a - bQ \), where \( Q \) is the total output in the industry. Assume \( a - c > 0 \) and \( b > 0 \).

(a) First suppose that there are \( n \) firms in the industry who have paid the fixed cost. Suppose that they compete as Cournot quantity setting oligopolists. How much will each firm produce? What will be the market price and the total quantity produced?

For each firm \( i \), we can solve the following maximization problem:

\[ \max_{q_i} [a - b(q_{-i} + q_i) - c]q_i \]

where \( q_{-i} = \sum_{j \neq i} q_j \). The FOC is:

\[ a - bq_{-i} - 2bq_i - c = 0 \]

\[ \Rightarrow q_i = \frac{a - bq_{-i} - c}{2b} \]

Since firms are symmetric, each firm produces the same amount in equilibrium:

\[ q_1 = q_2 = \ldots = q_n (:= q) \]

Using this property, we obtain:

\[ q = \frac{a - c}{(n + 1)b} \]
The total quantity is:

\[ Q = nq = \frac{n(a - c)}{(n + 1)b} \]

The market price is:

\[ p = a - bQ = \frac{a + nc}{n + 1} \]

(b) Now suppose that firms will exit the industry if their profit (net of fixed cost) is negative and that identical firms may enter if there are profits to be made. How many firms will enter? [Remember that your answer must be an integer].

For given \( n \), the profit of each firm becomes:

\[
\pi = (p - c)q - f \\
= \left( \frac{a + nc}{n + 1} - c \right) \frac{a - c}{(n + 1)b} - f \\
= \frac{1}{b} \left[ \frac{a - c}{(n + 1)} \right]^2 - f
\]

Thus, the equilibrium number of firms must be \( n^* \) such that:

\[
\frac{1}{b} \left[ \frac{a - c}{(n^* + 2)} \right]^2 - f < 0 \leq \frac{1}{b} \left[ \frac{a - c}{(n^* + 1)} \right]^2 - f
\]

If \( \frac{(a-c)^2}{b} < f \), then no firm would enter this market.

(c) What happens to the number of firms in the industry and prices as \( f \) becomes small? Give some economic intuition for your answer.

The inequality in (ii) illustrates that \( n^* \to \infty \) as \( f \to 0 \), which implies that the equilibrium price converges to \( c \). In words, more firms would enter as a fixed cost reduces, which eventually pushes down the market price to the competitive level, i.e., \( p = MC \).

3. Question 3 (30 points)

Two firms produce an identical good. The inverse demand curve for the good is \( P = 101 - X \), where \( X \) is the total quantity produced by the two firms. Firm 1 has a constant marginal cost \( 1 \) of producing the good. Firm 2 has a constant marginal cost \( 1 + c \) of producing the good, with \( 0 < c < 100 \).

(a) Suppose each firm \( i \) produces and sells \( x_i \) units of the good. Write down an expression for firm \( i \)'s profits (as a function of the output of each firm).
Given the output of the rival firm, each firm’s profit can be written as follows.

\[
\begin{align*}
\pi_1 &= (101 - x_1 - x_2)x_1 - x_1 \\
&= (100 - x_1 - x_2)x_1 \\
\pi_2 &= (101 - x_1 - x_2)x_2 - (1+c)x_2 \\
&= (100 - c - x_1 - x_2)x_2
\end{align*}
\]

(b) Suppose that each firm compete as quantity setting duopolists. What quantities will they produce, what is the market price and how much profit does each firm earn?

Each firm maximizes its own profit with respect to its own quantity. Therefore,

\[
\begin{align*}
\frac{\partial \pi_1}{\partial x_1} &= 100 - 2x_1 - x_2 = 0 \\
\frac{\partial \pi_2}{\partial x_2} &= 100 - c - 2x_2 - x_1 = 0,
\end{align*}
\]

which implies

\[
\begin{align*}
x_1 &\begin{cases} 
\frac{100-x_2}{2} & \text{if } x_2 < 100 \\
0 & \text{if } x_2 \geq 100
\end{cases} \\
x_2 &\begin{cases} 
\frac{100-c-x_1}{2} & \text{if } x_1 < 100 \\
0 & \text{if } x_1 \geq 100
\end{cases}
\end{align*}
\]

If we ignore the corner solutions, we obtain

\[
\begin{align*}
x_1^* &= \frac{100 + c}{3} \\
x_2^* &= \frac{100 - 2c}{3}.
\end{align*}
\]

Note that this is only valid for \( c \leq 50 \). If \( c > 50 \), we have \( x_2^* = 0 \), which implies that \( x_1^* = 50 \). That is, if \( c > 50 \), the second firm drops out of the market completely, because the price drops down so much that it doesn’t cover the marginal cost of firm 2.

The market price is determined by the demand function, so

\[
\begin{align*}
P^* &= 101 - (x_1^* + x_2^*) \\
&= \begin{cases} 
\frac{101+c}{3} & \text{if } c \leq 50 \\
51 & \text{if } c > 50
\end{cases}.
\end{align*}
\]
If \( c \leq 50 \), the firms’ profits are

\[
\pi_1^* = \left( \frac{100 + c}{3} \right)^2
\]

\[
\pi_2^* = \left( \frac{100 - 2c}{3} \right)^2.
\]

If \( c > 50 \), the profits are

\[
\pi_1^* = 2500, \pi_2^* = 0.
\]

(c) Suppose that firm 1 decides how much to produce first; firm 2 chooses only after observing firm 1’s choice. What quantities will they produce, what is the market price and how much profit does each firm earn?

Firm 1 knows that firm 2 will react to its production in an optimal way, which means that firm 2 will set

\[
x_2 = \begin{cases} 
\frac{100 - c - x_1}{2} & \text{if } x_1 \leq 100 - c \\
0 & \text{if } x_1 > 100 - c 
\end{cases}
\]

Firm 1 takes that into account when setting its quantity, therefore its profit is

\[
\pi_1 = \begin{cases} 
(100 - x_1 - \frac{100 - c - x_1}{2}) & \text{if } x_1 \leq 100 - c \\
(100 - x_1)x_1 & \text{if } x_1 > 100 - c 
\end{cases}
\]

If \( x_1 < 100 - c \), then by the first order condition, we get

\[
\frac{\partial \pi_1}{\partial x_1} = 100 - 2x_1 - \frac{100 - c - x_1}{2} = 50 - x_1 + \frac{1}{2}c = 0
\]

\[
\Rightarrow x_1^S = \frac{100 + c}{2} > x_1^*.
\]

Thus, the Stackelberg leader produces more than in the Cournot setting. Plugging \( x_1^S \) into the reaction function of firm 2,

\[
x_2^S = \frac{100 - c - x_1^S}{2} = \frac{100 - 3c}{4}.
\]

This works only for \( c \leq \frac{100}{3} \). The associated market price becomes

\[
P^S = 101 - x_1^S - x_2^S = \frac{104 + c}{4}.
\]
The profits are

\[
\pi_1^S = \frac{1}{2} \left( \frac{100 + c}{2} \right)^2 \\
\pi_2^S = \left( \frac{100 - 3c}{2} \right)^2.
\]

Note that firm 1 (resp. firm 2) has higher (resp. lower) profit than before.

If \( c > \frac{100}{3} \), firm 1 chooses \( x_1^S \geq 100 - c \) so that firm 2 produces 0. That is, \( x_2^S = \pi_2^S = 0 \) if \( c > \frac{100}{3} \). Since the monopoly quantity is 50, the firm 1’s optimal quantity become as follows

\[
x_1^S = \begin{cases} 
100 - c & \frac{100}{3} < c \leq 50 \\
50 & 50 < c
\end{cases}.
\]

Then, the profit becomes

\[
\pi_1^S = \begin{cases} 
(100 - c)e^{\frac{100}{3}} & \frac{100}{3} < c \leq 50 \\
2500 & 50 < c
\end{cases}.
\]